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Emergence of nonlinear Jeans-type instabilities for quasilinear wave equations

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Equations and goals

$$\partial_t^2 \varrho - \left(\frac{m^2 (\partial_t \varrho)^2}{(1+\varrho)^2} + 4(\hbar - m^2)(1+\varrho) \right) \Delta \varrho = F(t, \varrho, \partial_\mu \varrho)$$

where the nonlinear source terms are

$$F(t, \varrho, \partial_\mu \varrho) := \underbrace{\frac{2}{3}\varrho(1+\varrho)}_{\text{(i) self-increasing}} \underbrace{-\frac{1}{3}\partial_t \varrho}_{\text{(ii) damping}} + \underbrace{\frac{4}{3}\frac{(\partial_t \varrho)^2}{1+\varrho}}_{\text{(iii) Riccati}}$$

$$+ \underbrace{\left(m^2 \frac{(\partial_t \varrho)^2}{(1+\varrho)^2} + 4(\hbar - m^2)(1+\varrho) \right) q^i \partial_i \varrho}_{\text{(iv) convection}} - \mathrm{K}^{ij} \partial_i \varrho \partial_j \varrho.$$

$$\varrho|_{t=\mathbf{t}_0} = \beta + \psi(x^k) \quad \text{and} \quad \partial_t \varrho|_{t=\mathbf{t}_0} = \beta_0 + \psi_0(x^k), \quad \text{in } \{\mathbf{t}_0\} \times \mathbb{R}^n,$$

- (Goal) Find self-increasing blowup solutions.
- (Result) The solution blows up at the future end points of null geodesics and reaches arbitrarily large provided the data perturbations are sufficiently small (long wave feature!).

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After time transform $t \to \ln t$, the equation becomes:

$$\partial_t^2 \varrho - g^{ij}\partial_i\partial_j\varrho = \frac{2}{3t^2}\varrho(1+\varrho) - \frac{4}{3t}\partial_t\varrho + \frac{4}{3}\frac{(\partial_t\varrho)^2}{1+\varrho} + gq^i\partial_i\varrho$$
$$- \frac{1}{t^2}\mathrm{K}^{ij}(t,\varrho,\partial_\mu\varrho)\partial_i\varrho\partial_j\varrho, \quad \text{in } [t_0, t^\star) \times \mathbb{R}^n,$$
$$\varrho|_{t=t_0} = \beta + \psi(x^k) \quad \text{and} \quad \partial_t\varrho|_{t=t_0} = \beta_0 + \psi_0(x^k), \quad \text{in } \{t_0\} \times \mathbb{R}^n,$$

where

$$g^{ij} = g^{ij}(t,\varrho,\partial_t\varrho) := g(t,\varrho,\partial_t\varrho)\delta^{ij} = \left(m^2\frac{(\partial_t\varrho)^2}{(1+\varrho)^2} + 4(\kappa - m^2)\frac{1+\varrho}{t^2}\right)\delta^{ij}.$$

- Now focus on this equation!! A time transform $t \to e^t$ leads back to the previous equation.

Assumptions for simplifications

- $m^2 \leq \hbar$, $\beta \in (0, +\infty)$, $\beta_0 \in (0, +\infty)$, $t_0 \in (0, +\infty)$. $\psi \in C_0^1(\mathbb{R}^n)$ and $\psi_0 \in C_0^1(\mathbb{R}^n)$ are given positive-valued functions with $\mathrm{supp}\psi = B_1(0)$ and $\mathrm{supp}\psi_0 = B_1(0)$. $\mathtt{K}_{ij}$ be analytic functions in all their variables;
- The direction of convection to be a constant direction and it can be normalized $q^i = |q|\delta_1^i$ and $|q| \in (3, 100)$;
- $\hbar = \frac{1}{4}$.

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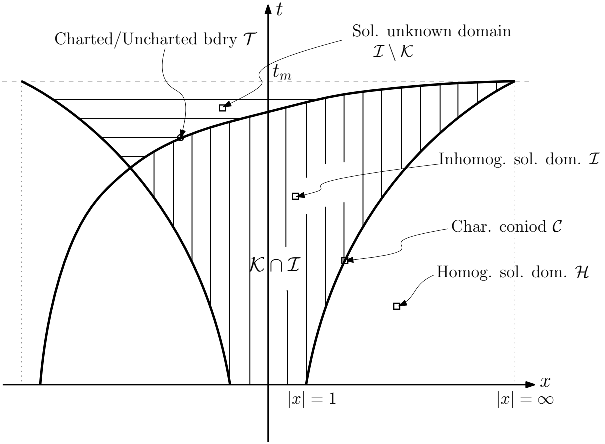
Main Theorem



$$\mathscr{I} := \left\{ (t, x) \in [t_0, t_m) \times \mathbb{R}^n \;\middle|\; |x| < 1 + \int_{t_0}^{t} \sqrt{g(y, f(y), f_0(y))} dy \right\},$$

$$\mathscr{H} := \left\{ (t, x) \in [t_0, t_m) \times \mathbb{R}^n \;\middle|\; |x| > 1 + \int_{t_0}^{t} \sqrt{g(y, f(y), f_0(y))} dy \right\},$$

$$\mathscr{C} := \left\{ (t, x) \in [t_0, t_m) \times \mathbb{R}^n \;\middle|\; |x| = 1 + \int_{t_0}^{t} \sqrt{g(y, f(y), f_0(y))} dy \right\}.$$

Main Theorem

Suppose $k \in \mathbb{Z}_{\frac{n}{2}+3}$, $A \in (0,2)$, $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$, $\mathtt{D}$ are constants depending on the initial data β and β_0, and Assumptions hold. Let $\psi \in C_0^1(\mathbb{R}^n)$ and $\psi_0 \in C_0^1(\mathbb{R}^n)$ be given functions with $\operatorname{supp}(\psi, \psi_0) = B_1(0)$, $f(\mathfrak{t})$ be the solution to key reference ODE.

Then there exist sufficiently small constants $\sigma_\star > 0$ and $\delta_\star > 0$, such that if the initial data satisfy

$$\|\psi\|_{H^k(B_1(0))} + \|\partial_i \psi\|_{H^k(B_1(0))} + \|\psi_0\|_{H^k(B_1(0))} \leq e^{-\frac{153}{\delta_0}} \sigma_0^2,$$

for any $\sigma_0 \in (0, \sigma_\star)$ and $\delta_0 \in (0, \delta_\star)$, then there exists a spacelike hypersurface $t = \mathfrak{T}(x, \delta_0)$ to the metric g satisfying

$$\mathcal{S}_{\delta_0} := \{(t,x) \in [t_0, t_m) \times \mathbb{R}^n \mid t = \mathfrak{T}(x, \delta_0)\} \subset \mathscr{I}, \qquad \lim_{a \to +\infty} \mathfrak{T}(a\delta_1^i, \delta_0) = t_m$$

$$\lim_{\delta_0 \to 0+} \mathfrak{T}(x, \delta_0) = \mathrm{b}_\uparrow(0) = t_m.$$

such that there is a solution $\varrho \in C^2(\mathscr{K} \cup \mathscr{H})$ to the main equation where $\mathscr{K} := \{(t,x) \in [t_0, t_m) \times \mathbb{R}^n \mid t < \mathfrak{T}(x, \delta_0)\}$ satisfying:

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Main Theorem (conti.)

- if we denote

$$\mathbf{1}_-(x^1) := 1 - C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \ (\searrow 1) \quad \text{and} \quad \mathbf{1}_+(x^1) := 1 + C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \ (\searrow 1), \quad \text{as } x^1 \to +\infty$$

then there are estimates for $(t, x) \in \mathscr{K} \cap \mathscr{I}$,

$$\begin{gathered}
\mathbf{1}_-(x^1) f_0\big(t_0 + \mathbf{1}_-(x^1)(t - t_0)\big) \leq \varrho_0(t, x) \leq \mathbf{1}_+(x^1) f_0\big(t_0 + \mathbf{1}_+(x^1)(t - t_0)\big) \\
-C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \big(1 + f\big(t_0 + \mathbf{1}_-(x^1)(t - t_0)\big)\big) \leq \varrho_i(t, x) \leq C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \big(1 + f\big(t_0 + \mathbf{1}_+(x^1)(t - t_0)\big)\big) \\
\mathbf{1}_-(x^1) f\big(t_0 + \mathbf{1}_-(x^1)(t - t_0)\big) \leq \varrho(t, x) \leq \mathbf{1}_+(x^1) f\big(t_0 + \mathbf{1}_+(x^1)(t - t_0)\big).
\end{gathered}$$

Moreover, ϱ_0 and ϱ reach the self increasing singularities at $p_m := (t_m, +\infty, 0, \cdots, 0)$:

$$\begin{gathered}
\lim_{\mathscr{K} \ni (t,x) \to p_m} \varrho = \lim_{\mathscr{K} \ni (t,x) \to p_m} f = +\infty, \\
\lim_{\mathscr{K} \ni (t,x) \to p_m} \varrho_0 = \lim_{\mathscr{K} \ni (t,x) \to p_m} f_0 = +\infty \quad \text{and} \quad \lim_{\mathscr{K} \ni (t,x) \to p_m} \varrho_i = 0.
\end{gathered}$$

- $\varrho \equiv f$ for $(t, x) \in \mathscr{H}$ where $\mathscr{H}$

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Main Theorem (conti.)

- the growth rate of ϱ can be estimated by

$$\varrho(t,x) \geq \mathbf{1}_-(x^1) f\left(t_0 + \mathbf{1}_-(x^1)(t-t_0)\right) > \mathbf{1}_-(x^1)\left(e^{\mathtt{C}(t_0+\mathbf{1}_-(x^1)(t-t_0))^{\frac{2}{3}}} - 1\right)$$

  and

$$\varrho(t,x) \leq \mathbf{1}_+(x^1) f\left(t_0 + \mathbf{1}_+(x^1)(t-t_0)\right) < \frac{3}{2}\left(\frac{1}{1+\frac{\mathtt{A}}{t_0+\mathbf{1}_+(x^1)(t-t_0)} + \mathtt{B}(t_0+\mathbf{1}_+(x^1)(t-t_0))^{\frac{2}{3}}} - 1\right)$$

  for all $(t,x) \in \mathscr{K} \cap \mathscr{I}$.

- if the initial data satisfy $\breve{\beta} := \frac{t_0 \mathring{f}_0}{1+\mathring{f}} - 1 > 0$, ϱ has an improved lower bound,

$$\varrho(t,x) \geq \mathbf{1}_-(x^1) f\left(t_0 + \mathbf{1}_-(x^1)(t-t_0)\right) > \mathbf{1}_-(x^1)\left(\frac{1+\mathring{f}}{\left(\frac{\beta_0 t_0^{\frac{4}{3}}}{1+\beta}\left(t_0 + \mathbf{1}_-(x^1)(t-t_0)\right)^{-\frac{1}{3}} - \breve{\beta}\right)^3} - 1\right)$$

  for all $(t,x^k) \in \mathscr{K} \cap \mathscr{I}$.

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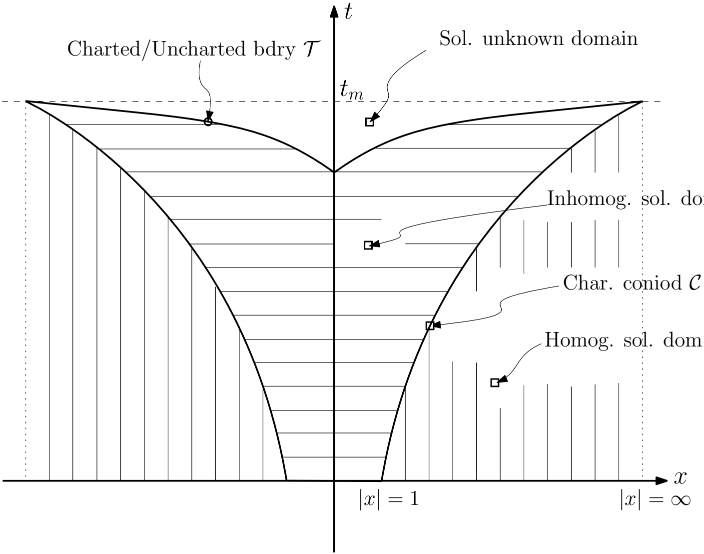
Generations



Backgrounds and motivations

A short version for the motivation

- In order to consider the nonlinear Jeans instability of the Euler–Poisson system and the Einstein–Euler.
- A toy model for above system. Neglecting the tidal force and the shear of the fluids, Euler–Poisson becomes this type of QNLW.
- Jeans instability characterizes the formation of nonlinear structures in the universe.

Classical Jeans instability (Static)

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$$\begin{cases} \partial_t \rho + \partial_i(\rho v^i) = 0 \\ \partial_t v^i + v^j \partial_j v^i + \dfrac{\partial^i p}{\rho} + \partial^i \phi = 0 \\ \delta^{ij}\partial_i\partial_j \phi = 4\pi G \rho \end{cases}$$

Euler – Poisson system

let $\rho = \underbrace{\rho_0}_{\text{constant.}} + \tilde{\rho}$, $v^i = \underbrace{v_0}_{0} + \tilde{v}$, $\phi = \underbrace{\phi_0}_{\nabla\phi_0 = 0} + \tilde{\phi}$, $\tilde{p} = c_s^2 \tilde{\rho}$

Contradicts Poisson.

① linearization $\Longrightarrow$

$$\begin{cases} \partial_t \tilde{\rho} + \rho_0 \partial_i \tilde{v}^i = 0 \\ \partial_t \tilde{v}^i + \dfrac{c_s^2}{\rho_0}\partial^i \tilde{\rho} + \partial^i \tilde{\phi} = 0 \\ \delta^{ij}\partial_i\partial_j \tilde{\phi} = 4\pi G \tilde{\rho} \end{cases} \xrightarrow{② \ \partial_i} \boxed{\partial_t^2 \tilde{\rho} - \underbrace{c_s^2}_{\text{Const.}} \delta^{ij}\partial_i\partial_j \tilde{\rho} - \underbrace{4\pi G \rho_0}_{\text{Const.}} \tilde{\rho} = 0}$$

$\Box\tilde{\rho} - k\tilde{\rho} = 0.$ $(k>0)$

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$$\boxed{\partial_t^2 \tilde{\rho} - c_s^2 \delta^{ij} \partial_i \partial_j \tilde{\rho} - 4\pi G \rho_0 \tilde{\rho} = 0}$$

$$\Big\Downarrow \text{ Fourier transform.}$$

$$\tilde{\rho}_k'' + (k^2 c_s^2 - 4\pi G \rho_0) \tilde{\rho}_k = 0 \Rightarrow \begin{cases} \tilde{\rho}_k \propto \exp(\pm i\omega(k) t) \\ \omega(k) = \sqrt{k^2 c_s^2 - 4\pi G \rho_0} \end{cases}$$

if $k^2 c_s^2 - 4\pi G \rho_0 < 0. \Leftrightarrow k < \frac{\sqrt{4\pi G \rho_0}}{c_s} =: k_J$

$$\Rightarrow \boxed{\tilde{\rho}_k \propto \exp(\pm|\omega| t)}$$

exponentially growth.

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Classical Jeans instability (expansion)

Expanding Newtonian Universe

$$\begin{cases} \rho = \rho_0(t) = \frac{1}{6\pi G t^2}, \quad v^i = v_0^i = \frac{2}{3t} x^i \quad \left(H(t) = \frac{2}{3t} \ \text{Hubble's law} \right) \\ \phi = \phi_0 = \frac{2}{3}\pi G \rho_0 |x|^2 \end{cases}$$

let $\rho = \rho_0 + \tilde{\rho}$, $v^i = v_0^i + \tilde{v}^i$, $\phi = \phi_0 + \tilde{\phi}$

$$\boxed{H = \frac{\dot{a}}{a}}$$

density contrast

① Lagrangian coord. (comoving with Hubble flow) : $x^i = a(t) q^i$

② Linearization Euler – Poisson

③ ∂_i(momentum conservation) ← Poisson eq. ← continuity eq.

let $e = \frac{\tilde{\rho}}{\rho_0}$ ⟹

$$\boxed{\partial_t^2 e + 2\underbrace{H}_{=\frac{4}{3t}}\dot{e} - \frac{c_s^2}{a^2}\delta^{ij}\partial_i\partial_j e - 4\pi G \rho_0 e = 0}$$

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$$\boxed{\partial_t^2 e + \frac{4}{3t}\dot{e} - \frac{c_s^2}{a^2}\delta^{ij}\partial_i\partial_j e - \frac{2}{3t^2}e = 0}$$

$\Big\Downarrow$ Fourier transform

$$e_k'' + \frac{4}{3t}e_k' + \left(\frac{c_s^2 k^2}{a^2} - \frac{2}{3t^2}\right)e_k = 0.$$

$\Big\Downarrow$ c_s small (pressure small)

$$e_k'' + \frac{4}{3t}e_k' - \frac{2}{3t^2}e_k = 0$$

$\Big\Downarrow$ Euler ODE

$$e_k = C_1 t^{-1} + C_2 t^{\frac{2}{3}} \Rightarrow |e| \sim t^{\frac{2}{3}}.$$

Our previous works

Try 1: Simple nonlinearity and Fuchsian formulations

$$\ddot{\varrho}+\frac{4}{3t}\dot{\varrho}-\tilde{\kappa}t^{-2\gamma+\frac{2}{3}}\Delta\varrho-\frac{2}{3t^2}\varrho=(\gamma-1)\tilde{\kappa}t^{-2\gamma+\frac{2}{3}}\frac{D^i\varrho D_i\varrho}{1+\varrho}. \tag{1}$$

Theorem

Suppose $s\in\mathbb{Z}_{\geq 3}$ and $\gamma>1$ are constants and $\mathring{\varrho}:=\varrho|_{t=1}$ and $\mathring{\varrho}_\mu:=(\partial_\mu\varrho)|_{t=1}$ $(\mu=0,\cdots,3)$. Let the initial data of the density satisfies an estimate

$$\left\|\mathring{\varrho}-\frac{\beta}{2}\right\|_{H^s(\mathbb{T}^3)}+\left\|\mathring{\varrho}_0-\frac{\beta}{3}\right\|_{H^s(\mathbb{T}^3)}+\|\mathring{\varrho}_i\|_{H^s(\mathbb{T}^3)}\leq\beta_0, \tag{2}$$

where $0<\beta<+\infty$ is any given constant and $\beta_0>0$ is a small enough constant. Then the solution of equation (1) satisfies

$$\frac{1}{4}\beta t^{\frac{2}{3}}\leq\varrho\leq\frac{3}{4}\beta t^{\frac{2}{3}}$$

for every $(t,\boldsymbol{q})$.

Methods

- Non-Fourier based method: Fuchsian formulations (energy method)
- Main difficulties:
  1. Find a compactified time $\tau \in [-1, 0)$ for physical time $t \in [t_0, \infty)$ such that $\tau = g(t)$.
  2. Select proper Fuchsian fields (similar define suitable energies)

Tool: Fuchsian formulations (KEY TOOL 1)

$$
\begin{aligned}
B^\mu \partial_\mu u =& \frac{1}{t}\mathbf{B}\mathbf{P}u + G \quad && \text{in } [-1,0)\times\mathbb{T}^n, \\
u =& u_0 && \text{on } \{-1\}\times\mathbb{T}^n.
\end{aligned}
$$

Some main assumptions of this system

1. $\mathbf{P}$ is a constant, symmetric projection operator (Pick up decay variables by $\mathbf{P}$).
2. $\frac{1}{\gamma_1}\mathbb{I} \le \mathring{B}^0 \le \frac{1}{\kappa}\mathring{\mathbf{B}} \le \gamma_2\mathbb{I}$ (Give right signs and determine dissipative effects).
3. $(B^\mu)^T = B^\mu, \quad [\mathbf{P},\mathbf{B}] = \mathbf{P}\mathbf{B} - \mathbf{B}\mathbf{P} = 0.$
4. $\mathbf{P}^\perp B^0(t,\mathbf{P}^\perp u)\mathbf{P} = \mathbf{P}B^0(t,\mathbf{P}^\perp u)\mathbf{P}^\perp = 0.$
5. $|\mathbf{P}^\perp[D_u B^0(t,u)(B^0)^{-1}\mathbf{B}\mathbf{P}u]\mathbf{P}^\perp|_{op} \le \alpha|t| + \beta|\mathbf{P}u|^2.$
6. ...... e.g., B^i, G somehow allows $\sim 1/t$ (extra condition) and $\sim 1/\sqrt{t}$......

(3, 4, 5 gives how the variables coupling to each other), and some regularity assumptions on the coefficients and remainders. Advantage: allow suitable coupling of variables.

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The Global Existence Theorem of the Cauchy problem of Fuchsian equations

Theorem (Oliynyk, 2016)

Suppose that $k \geq \frac{n}{2}+1$, $u_0 \in H^k(\mathbb{T}^n)$ and above conditions are fulfilled. Then there exists a $T_* \in (-1, 0)$, and a unique classical solution $u \in C^1([-1, T_*] \times \mathbb{T}^n)$ that satisfies $u \in C^0([-1, T_*], H^k) \cap C^1([-1, T_*], H^{k-1})$ and the energy estimate

$$\|u(t)\|^2_{H^k} - \int_{-1}^{t} \frac{1}{\tau} \|\mathbf{P} u\|^2_{H^k} d\tau \leq C e^{C(t+1)} (\|u(-1)\|^2_{H^k})$$

for all $-1 \leq t < T_*$, where $C = C(\|u\|_{L^\infty([-1,T_*),H^k)}, \gamma_1, \gamma_2, \kappa)$, and can be uniquely continued to a larger time interval $[T_0, T^*)$ for all $T^* \in (T_*, 0]$ provided $\|u\|_{L^\infty([-1,T_*),W^{1,\infty})} < \infty$.

This basic theorem has been generalized to more difficult cases and two parameter scales problems in the subsequent works by Oliynyk, L., Beyer, Olvera-Santamaría.

Intuitive toy model of Cauchy problem for Fuchsian system

- Rough idea: The following ODE dominated behaviors.
  Consider an ODE

$$\partial_t u = \frac{\beta}{t} u + (-t)^{-1+p} F(t), \quad \text{where} \quad 0 < p \leq 1, \beta > 0, t \in [-1, 0).$$

  Then

$$\partial_t \Big(u - \int_{-1}^{t} \frac{\beta}{s} u ds \Big) = (-t)^{-1+p} F(t).$$

  Integrating it yields

$$u - \int_{-1}^{t} \frac{\beta}{s} u ds \lesssim u_0 + 1 - (-t)^p.$$

  Further solving u leads to optimal decay estimates.
- The previous Theorem is obtained by adding conditions to make sure the Fuchsian system behaves like this toy model.

Try 2: Composite nonlinearities but with synchronizable sources

$$\boxed{\Box\varrho(x^\mu)+\frac{\mathfrak{a}}{t}\partial_t\varrho(x^\mu)-\frac{\mathfrak{b}}{t^2}\varrho(x^\mu)(1+\varrho(x^\mu))-\frac{\mathfrak{c}-\mathfrak{k}}{1+\varrho(x^\mu)}(\partial_t\varrho(x^\mu))^2=\mathfrak{k}F(t),}$$

$$\varrho|_{t=t_0}=\mathring{\varrho}(x^i)>0 \quad \text{and} \quad \partial_t\varrho|_{t=t_0}=\mathring{\varrho}_0(x^i)>0,$$

where $\Box:=\partial_t^2-\Delta_g=\partial_t^2-g^{ij}(t)\partial_i\partial_j$,

$$\mathfrak{a}>1, \quad \mathfrak{b}>0, \quad 1<\mathfrak{c}<3/2$$

$$g^{ij}(t):=\frac{m^2(\partial_t f(t))^2}{(1+f(t))^2}\delta^{ij} \quad \text{and} \quad F(t):=\frac{(\partial_t f(t))^2}{1+f(t)},$$

where $m\in\mathbb{R}$ is a given constant and $f(t)$ solves an ODE,

$$\boxed{\partial_t^2 f(t)+\frac{\mathfrak{a}}{t}\partial_t f(t)-\frac{\mathfrak{b}}{t^2}f(t)(1+f(t))-\frac{\mathfrak{c}}{1+f(t)}(\partial_t f(t))^2=0,}$$

$$f(t_0)=\mathring{f}>0 \quad \text{and} \quad \partial_t f(t_0)=\mathring{f}_0>0.$$

The solutions of ODEs (KEY TOOL 2)

$$\partial_t^2 f(t) + \frac{a}{t}\partial_t f(t) - \frac{b}{t^2} f(t)(1+f(t)) - \frac{c}{1+f(t)}(\partial_t f(t))^2 = 0,$$
$$f(t_0) = \mathring{f} > 0 \quad \text{and} \quad \partial_t f(t_0) = \mathring{f}_0 > 0.$$

Theorem

1. $t_\star \in [0, \infty)$ exists and $t_\star > t_0$;
2. there is a constant $t_m \in [t_\star, \infty]$, such that there is a unique solution $f \in C^2([t_0, t_m))$ to the ODE, and
$$\lim_{t\to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t\to t_m} f_0(t) = +\infty.$$
3. f satisfies upper and lower bound estimates,
$$1 + f(t) > \exp\left(\mathtt{C} t^{\frac{\bar{a}+\triangle}{2}} + \mathtt{D} t^{-1}\right) \qquad \text{for} \quad t \in (t_0, t_m);$$
$$1 + f(t) < \left(\mathtt{A} t^{\frac{\bar{a}-\triangle}{2}} + \mathtt{B} t^{\frac{\bar{a}+\triangle}{2}} + 1\right)^{-1} \qquad \text{for} \quad t \in (t_0, t_\star).$$

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$$\partial_t^2 f(t) + \frac{a}{t}\partial_t f(t) - \frac{b}{t^2} f(t)(1+f(t)) - \frac{c}{1+f(t)}(\partial_t f(t))^2 = 0,$$

$$f(t_0) = \mathring{f} > 0 \quad \text{and} \quad \partial_t f(t_0) = \mathring{f}_0 > 0.$$

Theorem

Furthermore, if the initial data satisfies $\mathring{f}_0 > \bar{a}(1+\mathring{f})/(\bar{c}t_0)$, then

4. $t_\star$ and $t^\star$ exist and finite, and $t_0 < t_\star < t^\star < \infty$;
5. there is a finite time $t_m \in [t_\star, t^\star)$, such that there is a solution $f \in C^2([t_0, t_m))$ to the ODE, and

$$\lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty.$$

6. the solution f has improved lower bound estimates, for $t \in (t_0, t_m)$,

$$(1+\mathring{f})\left(1 - \mathrm{E}t_0^{\bar{a}} + \mathrm{E}t^{\bar{a}}\right)^{1/\bar{c}} < 1 + f(t).$$

Pf: A series of differential inequalitites in the form of $\partial_t G < (>) \frac{k}{t} G + F$.

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The solutions to the PDEs

$$\boxed{\Box\varrho(x^\mu)+\frac{\mathfrak{a}}{t}\partial_t\varrho(x^\mu)-\frac{\mathfrak{b}}{t^2}\varrho(x^\mu)(1+\varrho(x^\mu))-\frac{\mathfrak{c}-\mathfrak{k}}{1+\varrho(x^\mu)}(\partial_t\varrho(x^\mu))^2=\mathfrak{k}F(t),}$$

$$\varrho|_{t=t_0}=\mathring{\varrho}(x^i)>0 \quad \text{and} \quad \partial_t\varrho|_{t=t_0}=\mathring{\varrho}_0(x^i)>0,$$

- Result: Self-increasing singularities and growth rate in f.

Method: the compactified time (KEY TOOL 3)

- Methods: Fuchsian formulations.
- The compactified time

$$\tau := -g(t) = -\exp\Big(-A\int_{t_0}^{t} \frac{f(s)(f(s)+1)}{s^2 f_0(s)}\,ds\Big)$$
$$= -\Big(1+\mathscr{b}B\int_{t_0}^{t} s^{\mathscr{a}-2} f(s)(1+f(s))^{1-\mathscr{c}}\,ds\Big)^{-\frac{A}{\mathscr{b}}} \in [-1,0),$$

Fuchsian fields

$$
\begin{aligned}
w(t,x^i) &:= \varrho(t,x^i) - f(t),\\
w_0(t,x^i) &:= \partial_t w(t,x^i) = \partial_t \varrho(t,x^i) - f_0(t),\\
w_i(t,x^i) &:= \partial_i w(t,x^i) = \partial_i \varrho(t,x^i).
\end{aligned}
$$

and

$$
\begin{aligned}
u(t,x^i) =& \frac{1}{f(t)} w(t,x^i),\\
u_0(t,x^i) =& \frac{1}{f_0(t)} w_0(t,x^i),\\
u_i(t,x^i) =& \frac{m}{1+f(t)} w_i(t,x^i).
\end{aligned}
$$

then

$$
\underline{u}(\tau,x^i) = u(g^{-1}(-\tau),x^i), \quad \underline{u_0}(\tau,x^i) = u_0(g^{-1}(-\tau),x^i)
$$
$$
\underline{u_i}(\tau,x^i) = u_i(g^{-1}(-\tau),x^i).
$$

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Singular/Regular-τ terms w/ special quantities and hiden relations (KEY TOOL 4)

Define a quantity

$$\chi(t) := \frac{t^{2-\mathcal{a}} f_0(t)}{(1+f(t))^{2-\mathcal{c}} f(t) g^{\frac{\mathcal{b}}{A}}(t)} = \frac{g^{-\frac{2\mathcal{b}}{A}}(t) t^{2(1-\mathcal{a})}}{Bf(t)(1+f(t))^{2(1-\mathcal{c})}}.$$

Then there is a function $\mathfrak{G} \in C^1([t_0, t_m))$, such that for $t \in [t_0, t_m)$,

$$\chi(t) = \frac{2\mathcal{b}B}{3-2\mathcal{c}} + \mathfrak{G}(t).$$

where $\lim_{t\to t_m} \mathfrak{G}(t) = 0$. Moreover, there is a constant $C_\chi > 0$ such that $0 < \chi(t) \leq C_\chi$ in $[t_0, t_m)$, and there are continuous extensions of χ and $\mathfrak{G}$ such that $\chi \in C^0([t_0, t_m])$ and $\mathfrak{G} \in C^0([t_0, t_m])$ by letting $\chi(t_m) := 2\mathcal{b}B/(3-2\mathcal{c})$ and $\mathfrak{G}(t_m) := 0$.

Define a quantity

$$\xi(t) := 1/[g(t)(1 + f(t))],$$

then $\xi \in C^1([t_0, t_m))$ and

$$\lim_{t \to t_m} \xi(t) = 0.$$

Moreover, there is a constant $C_\star > 0$, such that $0 < \xi(t) \leq C_\star$ for every $t \in [t_0, t_m)$, and there is a continuous extension of ξ such that $\xi \in C^0([t_0, t_m])$ by letting $\xi(t_m) := 0$.

Remark

$\chi(t)$ and $\xi(t)$ help distinguish the singular term $\frac{1}{\tau}\mathbf{BP}u$ and the regular term G in the Fuchsian system:

$$B^\mu \partial_\mu u = \frac{1}{\tau}\mathbf{BP}u + G.$$

Try 3& 4: Nonlinear gravitational instabilities (playground)

The dimensionless and normalized Euler–Poisson system

$$\partial_t \rho + \partial_i(\rho v^i) = 0,$$

$$\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0,$$

$$\partial_t s + v^i \partial_i s = 0,$$

$$\Delta \phi = \delta^{ij}\partial_i \partial_j \phi = 4\pi\rho.$$

The *equation of state* becomes

$$p = K e^s \rho^{\frac{4}{3}} + \mathfrak{p}, \quad \text{for } K \geq 0.$$

There is an exact solution on $(t, x^k) \in [t_0, \infty) \times \mathbb{R}^3$,

$$\mathring{\rho}(t) = \frac{\iota^3}{6\pi t^2}, \quad \mathring{p}(t) = K t^{-\frac{4}{3}} \delta_{kl} x^k x^l \mathring{\rho}^{\frac{4}{3}} + \mathfrak{p}, \quad \mathring{v}^i(t, x^k) = \frac{2}{3t} x^i,$$

$$\mathring{\phi}(t, x^k) = \frac{2}{3}\pi \mathring{\rho} \delta_{ij} x^i x^j = \frac{\iota^3}{9t^2}\delta_{ij} x^i x^j, \quad \mathring{s}(t, x^k) = \ln(t^{-\frac{4}{3}}\delta_{kl} x^k x^l)^{\mathrm{sgn}(1-\iota^3)},$$

Method: ODE from Key tool 2.

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We construct solutions

$$\rho(t) = (1+f(t))\mathring{\rho}(t) = \frac{\iota^3(1+f(t))}{6\pi t^2},$$

$$v^i(t,x^i) = \frac{2}{3t}x^i - \frac{f'(t)}{3(1+f(t))}x^i,$$

$$\phi(t,x^i) = \frac{2}{3}\pi\mathring{\rho}(1+f(t))|\boldsymbol{x}|^2 = \frac{\iota^3(1+f(t))|\boldsymbol{x}|^2}{9t^2},$$

$$s(t,x^k) = \ln\left(t^{-\frac{4}{3}}(1+f)^{\frac{2}{3}}\delta_{kl}x^k x^l\right)^{\mathrm{sgn}(1-\iota^3)}.$$

and the *density contrast* $\varrho(t) = f(t)$ where $|\boldsymbol{x}|^2 := \delta_{ij}x^i x^j$ and $f(t)$ is a solution of the following nonlinear ODE,

$$f''(t) + \frac{4}{3t}f'(t) - \frac{2}{3t^2}f(t)(1+f(t)) - \frac{4(f'(t))^2}{3(1+f(t))} = 0,$$

$$f|_{t=t_0} = \beta \quad \text{and} \quad f'|_{t=t_0} = 3(1+\beta)\gamma.$$

Moreover, the pressure becomes $p(t) = \frac{K\iota^4}{(6\pi)^{\frac{4}{3}}t^4}(1+f)^2\delta_{kl}x^k x^l$.

- Result: Self-increasing singularities.

Try 3& 4: Nonlinear gravitational instabilities (playground)

The dimensionless and normalized Euler–Poisson system

$$
\begin{aligned}
\partial_t \rho + \partial_i(\rho v^i) =& 0, \\
\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi =& \mathscr{D}^i(t, x^j, \rho, v^k, s, \phi), \\
\partial_t s + v^i \partial_i s =& \mathscr{S}(t, x^j, \rho, v^k, s, \phi), \\
\Delta \phi = \delta^{ij} \partial_i \partial_j \phi =& 4\pi\rho.
\end{aligned}
$$

EoS is

$$p = K e^s \rho^{\frac{4}{3}} \quad \text{for} \quad K > 0.$$

- $\mathscr{S}$ and $\mathscr{D}$ provide the synchronizable source like F.
- Transform to a type of Try 2;
- Self-increasing singularities.

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Eventually, we arrive at

$$\begin{aligned}\Box_g \hat{\varrho} + \Big(\frac{4}{3t} + \frac{\kappa f_0}{1+f}\Big)\partial_t \hat{\varrho} - \frac{2}{3t^2}\hat{\varrho}(1+\hat{\varrho}) - \frac{4(\partial_t \hat{\varrho})^2}{3(1+\hat{\varrho})} =& F_1,\\ \partial_t \nu + \frac{f_0}{3(1+f)}\nu\partial_\zeta \nu =& G_1,\end{aligned}$$

where the wave operator is

$$\Box_g := \partial_t^2 - g^{\zeta\zeta}\partial_\zeta^2 + 2g^{0\zeta}\partial_\zeta\partial_t,$$

$$g^{\zeta\zeta} := \frac{(2+\omega)(1-\iota^3)}{9t^2}\frac{(1+\hat{\varrho})^{\omega+1}}{(1+f)^\omega} - \frac{f_0^2}{9(1+f)^2}\nu^2, \quad g^{0\zeta} := \frac{f_0}{3(1+f)}\nu,$$

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Emergence of nonlinear Jean-type instabilities for QNLW
1. Main Theorems

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Equations and goals

$$\partial_t^2 \varrho - \left(\frac{m^2 (\partial_t \varrho)^2}{(1+\varrho)^2} + 4(\hbar - m^2)(1+\varrho) \right) \Delta \varrho = F(t, \varrho, \partial_\mu \varrho)$$

where the nonlinear source terms are

$$F(t, \varrho, \partial_\mu \varrho) := \underbrace{\frac{2}{3}\varrho(1+\varrho)}_{\text{(i) self-increasing}} \underbrace{-\frac{1}{3}\partial_t \varrho}_{\text{(ii) damping}} + \underbrace{\frac{4}{3}\frac{(\partial_t \varrho)^2}{1+\varrho}}_{\text{(iii) Riccati}}$$

$$+ \underbrace{\left(m^2 \frac{(\partial_t \varrho)^2}{(1+\varrho)^2} + 4(\hbar - m^2)(1+\varrho) \right) q^i \partial_i \varrho}_{\text{(iv) convection}} - \mathrm{K}^{ij} \partial_i \varrho \partial_j \varrho.$$

Data: $\varrho|_{t=\mathbf{t}_0} = \beta + \psi(x), \quad \partial_t \varrho|_{t=\mathbf{t}_0} = \beta_0 + \psi_0(x), \quad \text{in } \{\mathbf{t}_0\} \times \mathbb{R}^n,$

- (Goal) Find self-increasing blowup solutions.
- (Result) The solution blows up at the future end points of null geodesics and reaches arbitrarily large provided the data perturbations are sufficiently small (long wave feature!).

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After time transform $t \to \ln t$, the equation becomes:

$$\partial_t^2 \varrho - g^{ij}\partial_i\partial_j\varrho = \frac{2}{3t^2}\varrho(1+\varrho) - \frac{4}{3t}\partial_t\varrho + \frac{4}{3}\frac{(\partial_t\varrho)^2}{1+\varrho} + g q^i \partial_i\varrho$$
$$- \frac{1}{t^2}\mathtt{K}^{ij}(t,\varrho,\partial_\mu\varrho)\partial_i\varrho\partial_j\varrho, \quad \text{in } [t_0, t^\star) \times \mathbb{R}^n,$$

$$\text{Data: } \varrho|_{t=t_0} = \beta + \psi(x), \quad \partial_t\varrho|_{t=t_0} = \beta_0 + \psi_0(x), \quad \text{in } \{t_0\} \times \mathbb{R}^n,$$

where

$$g^{ij} = g^{ij}(t,\varrho,\partial_t\varrho) := g(t,\varrho,\partial_t\varrho)\delta^{ij} = \left(m^2\frac{(\partial_t\varrho)^2}{(1+\varrho)^2} + 4(\hbar - m^2)\frac{1+\varrho}{t^2}\right)\delta^{ij}.$$

- Now focus on this equation!! A time transform $t \to e^t$ leads back to the previous equation.

Assumptions for simplifications

- $m^2 \leq \mathcal{k}$, $\beta \in (0, +\infty)$, $\beta_0 \in (0, +\infty)$, $t_0 \in (0, +\infty)$. $\psi \in C_0^1(\mathbb{R}^n)$ and $\psi_0 \in C_0^1(\mathbb{R}^n)$ are given positive-valued functions with $\mathrm{supp}\psi = B_1(0)$ and $\mathrm{supp}\psi_0 = B_1(0)$. $\mathtt{K}_{ij}$ be analytic functions in all their variables;
- The direction of convection to be a constant direction and it can be normalized $q^i = |q|\delta_1^i$ and $|q| \in (3, 100)$;
- $\mathcal{k} = \frac{1}{4}$.

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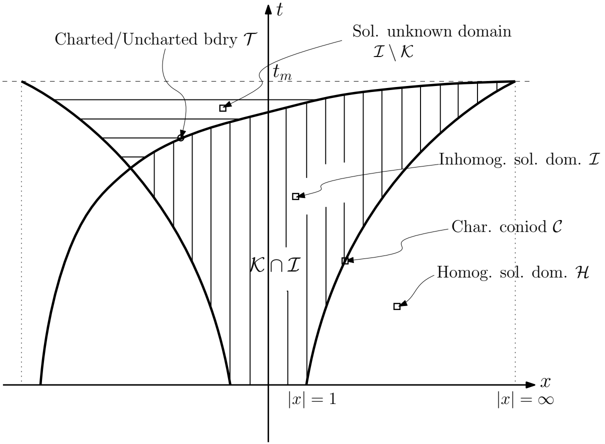
Main Theorem



$$\mathscr{I} := \left\{ (t,x) \in [t_0, t_m) \times \mathbb{R}^n \,\middle|\, |x| < 1 + \int_{t_0}^{t} \sqrt{g(y, f(y), f_0(y))}dy \right\},$$

$$\mathscr{H} := \left\{ (t,x) \in [t_0, t_m) \times \mathbb{R}^n \,\middle|\, |x| > 1 + \int_{t_0}^{t} \sqrt{g(y, f(y), f_0(y))}dy \right\},$$

$$\mathscr{C} := \left\{ (t,x) \in [t_0, t_m) \times \mathbb{R}^n \,\middle|\, |x| = 1 + \int_{t_0}^{t} \sqrt{g(y, f(y), f_0(y))}dy \right\}.$$

Main Theorem

Suppose $k \in \mathbb{Z}_{\frac{n}{2}+3}$, $A \in (0,2)$, $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$, $\mathtt{D}$ are constants depending on the initial data β and β_0, and Assumptions hold. Let $\psi \in C_0^1(\mathbb{R}^n)$ and $\psi_0 \in C_0^1(\mathbb{R}^n)$ be given functions with $\mathrm{supp}(\psi, \psi_0) = B_1(0)$, $f(\mathfrak{t})$ be the solution to key reference ODE.
Then there exist sufficiently small constants $\sigma_\star > 0$ and $\delta_\star > 0$, such that if the initial data satisfy

$$\|\psi\|_{H^k(B_1(0))} + \|\partial_i \psi\|_{H^k(B_1(0))} + \|\psi_0\|_{H^k(B_1(0))} \leq e^{-\frac{153}{\delta_0}} \sigma_0^2,$$

for any $\sigma_0 \in (0, \sigma_\star)$ and $\delta_0 \in (0, \delta_\star)$, then there exists a spacelike hypersurface $t = \mathfrak{T}(x, \delta_0)$ to the metric g satisfying

$$\mathcal{S}_{\delta_0} := \{(t,x) \in [t_0, t_m) \times \mathbb{R}^n \mid t = \mathfrak{T}(x, \delta_0)\} \subset \mathcal{I}, \quad \lim_{a \to +\infty} \mathfrak{T}(a\delta_1^i, \delta_0) = t_m$$

$$\lim_{\delta_0 \to 0+} \mathfrak{T}(x, \delta_0) = \mathrm{b}_\uparrow(0) = t_m.$$

such that there is a solution $\varrho \in C^2(\mathcal{K} \cup \mathcal{H})$ to the main equation where $\mathcal{K} := \{(t,x) \in [t_0, t_m) \times \mathbb{R}^n \mid t < \mathfrak{T}(x, \delta_0)\}$ satisfying:

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Main Theorem (conti.)

- if we denote

$$\mathbf{1}_-(x^1) := 1 - C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \;(\searrow 1) \quad \text{and} \quad \mathbf{1}_+(x^1) := 1 + C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \;(\searrow 1), \quad \text{as } x^1 \to +\infty$$

then there are estimates for $(t, x) \in \mathscr{K} \cap \mathscr{I}$,

$$\begin{gathered}
\mathbf{1}_-(x^1) f_0\big(t_0 + \mathbf{1}_-(x^1)(t - t_0)\big) \leq \varrho_0(t,x) \leq \mathbf{1}_+(x^1) f_0\big(t_0 + \mathbf{1}_+(x^1)(t - t_0)\big) \\
-C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \big(1 + f\big(t_0 + \mathbf{1}_-(x^1)(t - t_0)\big)\big) \leq \varrho_i(t,x) \leq C\sigma_0^2 e^{-\frac{103}{\delta_0}} e^{-\frac{x^1}{2}} \big(1 + f\big(t_0 + \mathbf{1}_+(x^1)(t - t_0)\big)\big) \\
\mathbf{1}_-(x^1) f\big(t_0 + \mathbf{1}_-(x^1)(t - t_0)\big) \leq \varrho(t,x) \leq \mathbf{1}_+(x^1) f\big(t_0 + \mathbf{1}_+(x^1)(t - t_0)\big).
\end{gathered}$$

Moreover, ϱ_0 and ϱ reach the self increasing singularities at $p_m := (t_m, +\infty, 0, \cdots, 0)$:

$$\begin{gathered}
\lim_{\mathscr{K} \ni (t,x) \to p_m} \varrho = \lim_{\mathscr{K} \ni (t,x) \to p_m} f = +\infty, \\
\lim_{\mathscr{K} \ni (t,x) \to p_m} \varrho_0 = \lim_{\mathscr{K} \ni (t,x) \to p_m} f_0 = +\infty \quad \text{and} \quad \lim_{\mathscr{K} \ni (t,x) \to p_m} \varrho_i = 0.
\end{gathered}$$

- $\varrho \equiv f$ for $(t, x) \in \mathscr{H}$ where $\mathscr{H}$

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Main Theorem (conti.)

- the growth rate of ϱ can be estimated by

$$\varrho(t,x) \geq \mathbf{1}_{-}(x^1) f\left(t_0 + \mathbf{1}_{-}(x^1)(t-t_0)\right) > \mathbf{1}_{-}(x^1)\left(e^{\mathtt{C}(t_0+\mathbf{1}_{-}(x^1)(t-t_0))^{\frac{2}{3}}} - 1\right)$$

and

$$\varrho(t,x) \leq \mathbf{1}_{+}(x^1) f\left(t_0 + \mathbf{1}_{+}(x^1)(t-t_0)\right) < \frac{3}{2}\left(\frac{1}{1+\frac{\mathtt{A}}{t_0+\mathbf{1}_{+}(x^1)(t-t_0)} + \mathtt{B}(t_0+\mathbf{1}_{+}(x^1)(t-t_0))^{\frac{2}{3}}} - 1\right)$$

for all $(t,x) \in \mathscr{K} \cap \mathscr{I}$.

- if the initial data satisfy $\breve{\beta} := \frac{t_0 \mathring{f}_0}{1+\mathring{f}} - 1 > 0$, ϱ has an improved lower bound,

$$\varrho(t,x) \geq \mathbf{1}_{-}(x^1) f\left(t_0 + \mathbf{1}_{-}(x^1)(t-t_0)\right) > \mathbf{1}_{-}(x^1)\left(\frac{1+\mathring{f}}{\left(\frac{\beta_0 t_0^{\frac{4}{3}}}{1+\beta}\left(t_0+\mathbf{1}_{-}(x^1)(t-t_0)\right)^{-\frac{1}{3}} - \breve{\beta}\right)^3} - 1\right)$$

for all $(t,x^k) \in \mathscr{K} \cap \mathscr{I}$.

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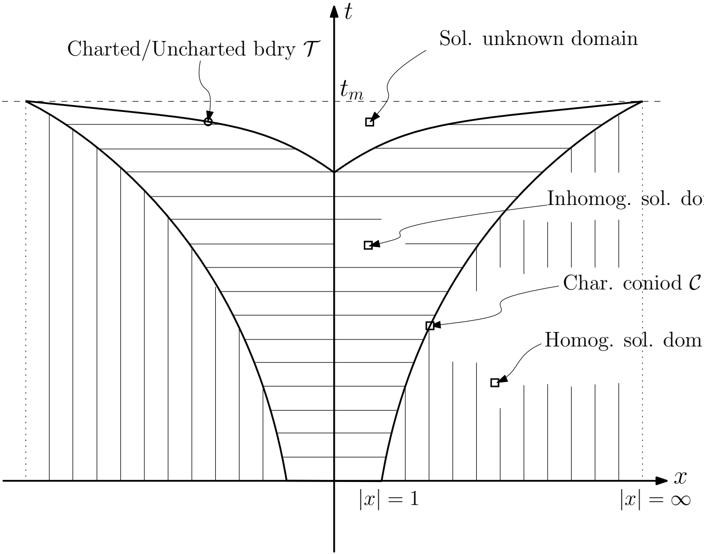
Generations



Emergence of nonlinear Jean-type instabilities for QNLW
2. Ideas of the proofs

Ideas and Fuchsian direction

- Basic direction: the Fuchsian method!
- Require time compactifications. How?
  Idea: Intro. a compactified time like “Try 2”?
  Difficulty: Fail! Since there is no synchronized term (synchronizing source term synchronize the blowup time to 0), and it is high possible that the solution blows up at different time (if it blows up!). The compactified time works only if the blow up time can be synchronized and the perturbations do not change the blowup times (if blwoup at infinity, it may still work)

Compactified time

- Overcome diff.: Intro. two compactified time:
  (1) for the reference solution (sol. to ref. ODE), use "try 2" compactified time;

$$\tau = \mathfrak{g}(\mathfrak{t}) = -\Big(1 + \frac{2}{3}B\int_{t_0}^{\mathfrak{t}} s^{-\frac{2}{3}} f(s)(1+f(s))^{-\frac{1}{3}} ds\Big)^{-\frac{3A}{2}} \in [-1, 0).$$

  It synchronizes the blowup time of the reference solution. However, the perturbations may not blowup at this time, blowup time may deviate it.

- In order to be comparable (this may not hold!), we intro the compactified time analogue to this

$$\tau = g(t, x^i) = -\Big(1+\frac{2}{3}B\int_{t_0}^{t} s^{-\frac{2}{3}} \varrho(s, x^i)(1+\varrho(s, x^i))^{-\frac{1}{3}} ds\Big)^{-\frac{3A}{2}} \in [-1, 0)$$

- Wrong compactified time leads wrong structures and fails. It is crucial how to choose it. Need guess and experiments!

ODE equivalence of the compactification

The compacitified time can be reexpressed in terms of two ODEs:

$$\partial_t g(t,x^i) = \frac{AB\varrho(t,x^i)\left(-g(t,x^i)\right)^{\frac{2}{3A}+1}}{t^{\frac{2}{3}}(\varrho(t,x^i)+1)^{\frac{1}{3}}},$$
$$g(t_0,x^i) = -1.$$

and

$$\partial_{\mathfrak{t}}\mathfrak{g}(\mathfrak{t}) = -A\mathfrak{g}(\mathfrak{t})\frac{f(\mathfrak{t})(f(\mathfrak{t})+1)}{\mathfrak{t}^2 f_0(\mathfrak{t})} = \frac{ABf(\mathfrak{t})(-\mathfrak{g}(\mathfrak{t}))^{1+\frac{2}{3A}}}{\mathfrak{t}^{\frac{2}{3}}(1+f(\mathfrak{t}))^{\frac{1}{3}}},$$
$$\mathfrak{g}(t_0) = -1.$$

- The coordinate transform requires the knowing of Jacobians, these ODEs provide the Jacobian and determines how the coordinate transform develops.
- They provide some hiden identities.

The first coordinate transform

We express the main equation to a singular hyperbolic system (1st order) in terms of (τ, ζ) given by

$$\tau = g(t, x^i) \quad \text{and} \quad \zeta^i = x^i$$

Its inverse transformation denote

$$t = \mathrm{b}(\tau, \zeta^i) \quad \text{and} \quad x^i = \zeta^i$$

and satisfies a ODE (Why? Since it is Fuchsianable)

$$\partial_\tau \mathrm{b}(\tau, \zeta^i) = \frac{\mathrm{b}^{\frac{2}{3}}(\tau, \zeta^i)(1 + \underline{\varrho}(\tau, \zeta^i))^{\frac{1}{3}}}{AB\underline{\varrho}(\tau, \zeta^i)\,(-\tau)^{\frac{2}{3A}+1}},$$

$$\mathrm{b}(-1, \zeta^i) = t_0$$

- We do not give the coordinate transform directly but give it by an evolution equation (similar to the wave coordinates, perturbed Lagrangian coordinates, etc.)
- b and $\mathbf{b}_\zeta := \partial_\zeta \mathbf{b}$ become unknown variables since they describe the coordinate transform and this transform has been solved from an equation.

Singular symmetric hyperbolic system

- Intro. perturbation variables: e.g. $u(\tau, \zeta^k) = \frac{\underline{\varrho}(\tau,\zeta^k) - \underline{f}(\tau)}{\underline{f}(\tau)}$
- Using a lot of hiden relations derived from the reference ODE and the quantities χ and ξ in "try 2" we can have a singular symmetric hyperbolic equation (similar to "try 2").
- Comparing with "Try 2", this method has already lead to the Fuchsian system and it is done! However, now it can not be achieved.

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Lemma

$$\mathbf{A}^0\partial_\tau U+\frac{1}{A\tau}\mathbf{A}^i\partial_{\zeta^i}U=\frac{1}{A\tau}\mathbf{A}U+\mathbf{F},$$

where $U:=\left(u_0,u_j,u,\mathcal{B}_l,z\right)^T$, $\mathbf{F}=\left(\mathfrak{F}_{u_0},\mathfrak{F}_{u_j},\mathfrak{F}_u,\mathfrak{F}_{\mathcal{B}_i},\mathfrak{F}_z\right)^T$,

$$\mathbf{A}^0=\begin{pmatrix}1 & \mathscr{R}^j & 0 & 0 & 0\\ \mathscr{R}_k & (S+\mathscr{L})\delta^j_k & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & \delta^l_s & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},\qquad \mathbf{A}^i=\begin{pmatrix}0 & H^{ij} & 0 & 0 & 0\\ H^i_k & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & q^i\end{pmatrix},$$

$$\mathbf{A}=\begin{pmatrix}-\frac{14}{3}+\mathscr{Z}_{11} & -4\hbar q^j+\mathscr{Z}_{12}^j & 8+\mathscr{Z}_{13} & 0 & -8+\mathscr{Z}_{15}\\ 0 & (4\hbar+\mathscr{Z}_{22})\delta^j_k & 0 & (24\hbar+\mathscr{Z}_{24})\delta^l_k & 0\\ -8+\mathscr{Z}_{31} & 0 & \frac{40}{3}+\mathscr{Z}_{33} & 0 & -16+\mathscr{Z}_{35}\\ 0 & (\frac{2}{3}+\mathscr{Z}_{42})\delta^j_s & 0 & (\frac{2}{3}+\mathscr{Z}_{44})\delta^l_s & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

Difficulty: $\mathbf{A}$ can not be positive definite whatever you do!

Two revisions

(1) The second coordinate transform: the tilted coordinate

$$\tilde{\tau} = \tilde{\tau}(\tau, \zeta^k) = \tau \quad \text{and} \quad \tilde{\zeta}^i = \tilde{\zeta}^i(\tau, \zeta^k) = \frac{\mathrm{ac}^i}{A} \ln(-\tau) + \zeta^i,$$

- Motivations: Expand the "null infinity" (not precisely) and upright a timelike direction (close to null) to be the time axis. since our analysis can only work in this "closed to null" domain.
- From the equation point of view, (1) generate more terms in $\frac{1}{A\tau}\mathbf{A}^i$ and will help compensate $\frac{1}{A\tau}\mathbf{A}$ to achieve the positive definiteness.
- From the geometric point of view, they tilt the characteristic conoid and expand the "near-null" domain.

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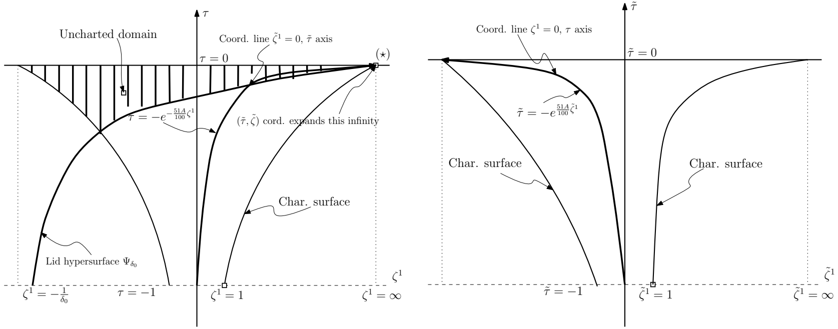


(2) rescale all the variables by spatial factors, e.g., $\mu(\tilde{\zeta}^1) := \sigma_0 e^{-\frac{153}{\delta_0}} e^{-51\tilde{\zeta}^1}$ and the variable, e.g., becomes

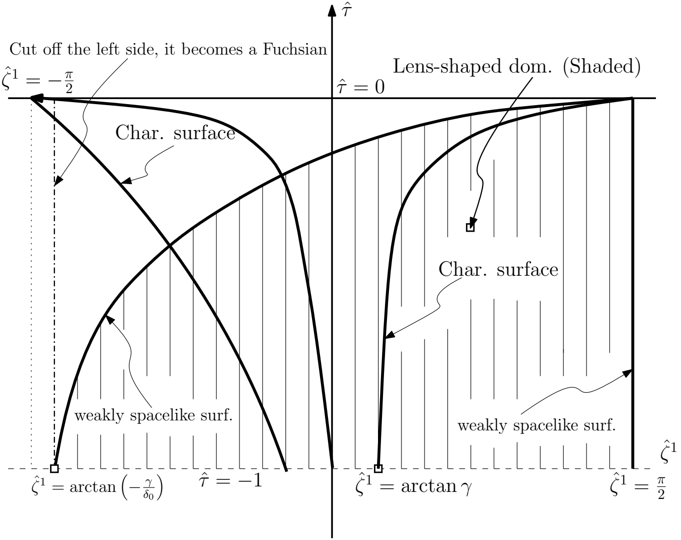
$$u_0(\tilde{\tau}, \tilde{\zeta}) = \frac{1}{\sigma_0 e^{-\frac{153}{\delta_0}} e^{-51\tilde{\zeta}^1}} \widetilde{u_0}(\tilde{\tau}, \tilde{\zeta})$$

- Motivation: Spatial factors like μ will separate a new singular remainder term $\frac{1}{A\tau}\mathbf{A}_{\text{remainder}}U$ from $\frac{1}{A\tau}\mathbf{A}^i\partial_i U$, and $\frac{1}{A\tau}\mathbf{A}_{\text{remainder}}U$ compensate $\frac{1}{A\tau}\mathbf{A}U$ to obtain a positive definite singular lower order term (consists with the Fuchsian).
- Defect: $\mu \sim e^{-51\tilde{\zeta}^1}$ introduce infinities to the equation as $\tilde{\zeta}^1 \to -\infty$. Break the structures.
- Idea to overcome: Revise the equation by cutoff function ϕ such that the infinities vanish. However, the equation fails to equivalent to the original equation due to the revision.

$$\phi \in C^\infty\big(\mathbb{R}; [0,1]\big), \quad \phi|_{[-\delta_0^{-1}, +\infty)} = 1 \quad \text{and} \quad \text{supp}\phi \subset [-2\delta_0^{-1}, +\infty) \subset \mathbb{R}.$$

How to recover the solution of the original one?

- To recover original solution, only use the lens-shaped domain (determination domain, see Fig. to explain)



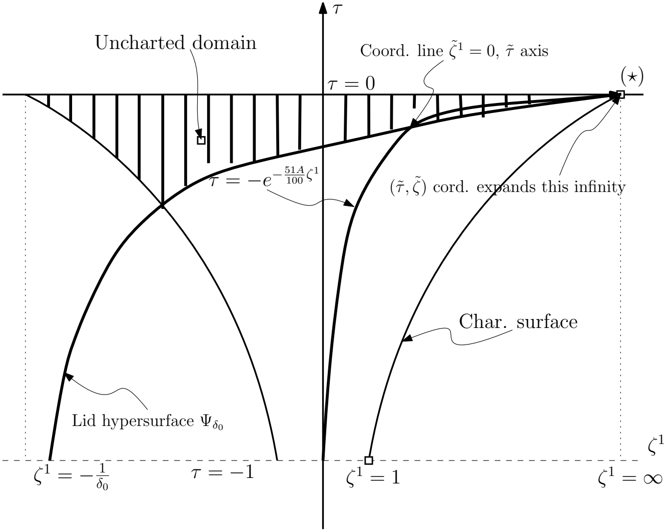
The revised system becomes a Fuchsian system by compactifying space

The third coordinate transform (compactifying the space)

$$\hat{\tau} = \tilde{\tau} \in [-1, 0) \quad \text{and} \quad \hat{\zeta}^i = \arctan(\gamma \tilde{\zeta}^i) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- $\mathbb{R}^n$ becomes $\mathbb{T}^n$, a closed manifold which is required by the Fuchsian analysis.
- After this coordinate transform, we have Fuchsian formulation and can derive the global existence and stability result for this revised system.
- Using determination domain obtain the main theorem.

This talk was given before the article's release, so some statements and notations may differ from the final version. However, the core ideas remain consistent.



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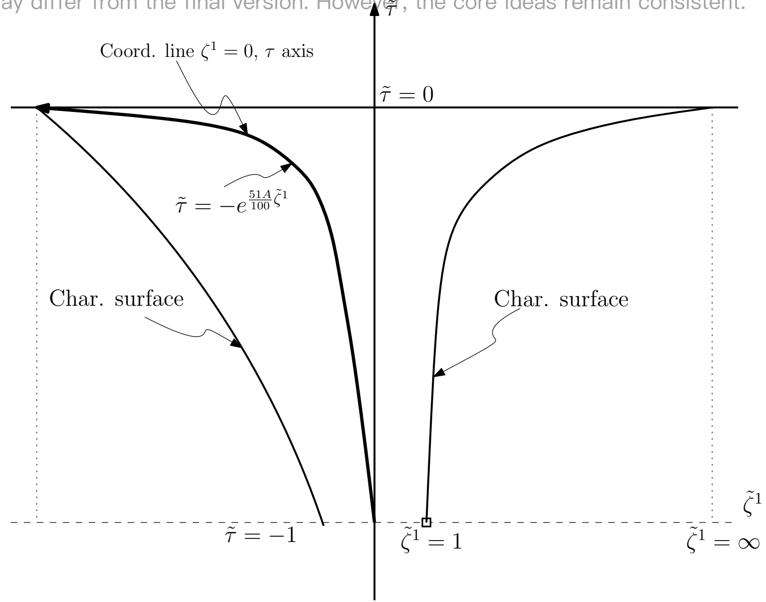

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Thank you
for your attention!