Future global existence and stability of de Sitter-like solutions to the Einstein–Yang–Mills equations in spacetime dimensions $n \geq 4$

(joint with Todd A. Oliynyk and Jinhua Wang)

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Backgrounds

Einstein–Yang–Mills equations with $\Lambda > 0$

- Einstein–Yang–Mills (EYM) equations with a positive cosmological constant $\Lambda > 0$ is given by

$$\tilde{G}_{ab} + \Lambda \tilde{g}_{ab} = \tilde{T}_{ab},$$
$$\tilde{D}^a \tilde{F}_{ab} = 0,$$
$$\tilde{D}_{[a} \tilde{F}_{bc]} = 0,$$

  where $\tilde{G}_{ab} = \tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab}$ is the Einstein tensor of the metric $\tilde{g}_{ab}$ and the stress energy tensor of a Yang–Mills field is defined by

$$\tilde{T}_{ab} = \tilde{F}_a{}^c \cdot \tilde{F}_{bc} - \frac{1}{4}\tilde{g}_{ab}\tilde{F}^{cd} \cdot \tilde{F}_{cd}.$$

- Let G denote a compact and connected Lie group with Lie algebra $\mathcal{G}$. Due to the compactness, we lose no generality in taking G to be a matrix group.

- Given a n-dimensional, connected Lorentzian manifold $(\widetilde{\mathcal{M}}^n, \tilde{g}_{ab})$, a connection $\tilde{\omega}$ on a G-principal bundle over $\widetilde{\mathcal{M}}^n$ can be expressed in a gauge as a local $\mathcal{G}$-valued 1-form $\tilde{A}_a$ on $\widetilde{\mathcal{M}}^n$, which is referred to as a gauge potential. The curvature of the connection $\tilde{\omega}$ is then determined locally by the $\mathcal{G}$-valued 2-form $\tilde{F}_{ab}$ on $\widetilde{\mathcal{M}}^n$ defined by

$$\tilde{F}_{ab} = \tilde{\nabla}_a \tilde{A}_b - \tilde{\nabla}_b \tilde{A}_a + [\tilde{A}_a, \tilde{A}_b],$$

where $[\cdot, \cdot]$ is the Lie bracket on $\mathcal{G}$, i.e. the matrix commutator bracket, and $\tilde{\nabla}_a$ is the covariant derivative associated to $\tilde{g}_{ab}$. We recall also that the Yang–Mills curvature $\tilde{F}_{ab}$ automatically satisfies the Bianchi identities

$$\tilde{D}_{[a}\tilde{F}_{bc]} = 0,$$

where $\tilde{D}_a = \tilde{\nabla}_a + [\tilde{A}_a, \cdot]$ denotes the gauge covariant derivative of a $\mathcal{G}$-valued tensor.

de Sitter spacetime

- de Sitter spacetime $(\widetilde{\mathcal{M}}^n, \underline{\tilde{g}}_{ab})$ is obtained by equipping $\widetilde{\mathcal{M}}^n = \mathbb{R} \times \mathbb{S}^{n-1}$ with the de Sitter metric defined by

$$\underline{\tilde{g}}_{ab} = -(d\tau)_a(d\tau)_b + H^2 \cosh^2(H^{-1}\tau)\underline{h}_{ab}$$

where the constant H is determined by

$$H = \sqrt{\frac{(n-2)(n-1)}{2\Lambda}},$$

$\underline{h}_{ab}$ is the standard metric on $\mathbb{S}^{n-1}$, and τ is a Cartesian coordinate function on $\mathbb{R}$.

Conformal picture of de Sitter

Introduce the conformal time function t via

$$t = \frac{1}{H}\left(\frac{\pi}{2} - \mathrm{gd}(H^{-1}\tau)\right) \quad \text{i.e.} \quad \tau = H\,\mathrm{gd}^{-1}\left(\frac{\pi}{2} - Ht\right)$$

where $\mathrm{gd}(x)$, known as the Gudermannian function, is defined by

$$\mathrm{gd}(x) = \int_0^x \frac{1}{\cosh s} ds = \arctan\big(\sinh(x)\big), \quad x \in \mathbb{R},$$
$$\mathrm{gd}^{-1}(x) = \int_0^x \frac{1}{\cos t} dt = \operatorname{arctanh}(\sin x), \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Then, the de Sitter metric can be written as

$$\tilde{\underline{g}}_{ab} = e^{2\Psi}\underline{g}_{ab} \quad \text{where} \quad \Psi = -\ln\left(\frac{\sin(Ht)}{H}\right) \sim -\ln(t)$$

and

$$\underline{g}_{ab} = -H^2(dt)_a(dt)_b + \underline{h}_{ab}$$

is the conformal de Sitter metric.

Important remark on the conformal spacetime

- The pair $(\mathcal{M}^n = \left(0, \frac{\pi}{H}\right) \times \Sigma,\ \underline{g}_{ab})$ defines a spacetime that is conformal to the de Sitter spacetime where, by construction, future timelike infinity of the de Sitter spacetime is mapped to the boundary component $\{0\} \times \Sigma$ of $\mathcal{M}^n$.
- Statement of Main Theorem is in terms of the physical picture since it is good for local existence in the temporal gauge. Main proof is in terms of the conformal picture since it is good for global stability by Fuchsian formulations.

Very brief reviews on de Sitter-like stability

Dimensional dependent methods

- H. Friedrich (1986), the vacuum dS_4 stability (Conformal method with conformal invariants).
- M. Anderson (2005), the vacuum dS_n stability for any even $n \geq 4$ (Conformal, Fefferman-Graham tensor).
- H. Friedrich (1991), dS_4-like stability for the Einstein-Maxwell-Yang-Mills field.

Dimensional dependent methods

- H. Ringström (2008), the nonlinear dS_n-like stability for any $n \geq 4$ for Einstein-scalar field.
- C. Svedberg (2011), Rinström 's result was extended to Einstein-Maxwell-Scalar field (Lorentz gauge).

Brief reviews on the FLRW stability for Einstein-Euler system

de Sitter-like stability

- I. Rodnianski, J. Speck, C. Lübble, J. Kroon and M. Hadžić (2013,2014,2015), NL stability of FLRW with $\Lambda > 0$ and linear EOS for $0 < K < 1/3$, $K = 1/3$ and $K = 0$ respectively.
- T. Oliynyk (2016), NL stability of FLRW with $\Lambda > 0$ and linear EOS for $0 < K \leq 1/3$ (using Fuchsian type system).
- T. Oliynyk, C. L., P. Lefloch and C. Wei (2017-2020), NL stability of FLRW with $\Lambda > 0$ for various nonlinear EOS and related cosmological Newtonian limit problems (using Fuchsian type system).

Other non-de Sitter-like stability: L. Andersson, J. Wang, D. Fajman, et al for FLRW-like ($\Lambda = 0$) to Einstein-Klein-Gordon, Vlasov, ...

Variables and gauge choices in the Main Theorem

- Note in the main proof of the global stability, we use the conformal picture and use slightly changed varibles and gauges.
- Decompose the physical Yang–Mills field by

$$\tilde{E}_b = \tilde{h}^a{}_b \tilde{F}_{ap} \tilde{T}^p \quad \text{and} \quad \tilde{H}_{db} = \tilde{h}^c{}_d \tilde{F}_{ca} \tilde{h}^a{}_b.$$

where $\tilde{T}^a = (-\tilde{\lambda})^{-\frac{1}{2}} \tilde{g}^{ab}(d\tau)_b$ and $\tilde{h}^c{}_d = \delta^c{}_d + \tilde{T}^c \tilde{T}_d$.

- Temporal gauge: $\tilde{A}_a \tilde{T}^a = 0$.
- Wave gauge: $\tilde{Z}^a = 0$, where (this form can kill the "bad" terms in the conformal picture)

$$\tilde{Z}^a = \tilde{X}^a - 2(\underline{\tilde{g}}^{ac} - \tilde{g}^{ac})(d\Psi)_c + n(\tilde{g}^{ac} - \underline{\tilde{g}}^{ac})(d\Psi)_c - (\tilde{g}^{fe} - \underline{\tilde{g}}^{fe})\underline{\tilde{g}}^{ac}\underline{\tilde{g}}_{fe}(d\Psi)_c$$

and

$$\tilde{X}^a = -\underline{\tilde{\nabla}}_e \tilde{g}^{ae} + \frac{1}{2}\tilde{g}^{ae}\tilde{g}_{df}\underline{\tilde{\nabla}}_e \tilde{g}^{df}.$$

Here, $\underline{\tilde{\nabla}}_e$ denotes the covariant derivative associated to the de Sitter metric $\underline{\tilde{g}}_{ab}$ and Ψ is the conformal factor.

Main Theorem (L-Oliynyk-Wang, JEMS, to appear)

Suppose $\Lambda > 0$, $s \in \mathbb{Z}_{>\frac{n+1}{2}}$, and the H^s-initial data satisfy the constraint equations. Then there exists a constant $\sigma > 0$ such that if the initial data satisfy smallness condition

$$\|(\tilde{g}^{ab}(0) - \underline{\tilde{g}}^{ab}(0), \underline{\tilde{\nabla}}_d \tilde{g}^{ab}(0), \tilde{A}_a(0), \tilde{E}_a(0), \tilde{H}_{ab}(0))\|_{H^s} \leq \sigma,$$

then there exists a unique solution $(\tilde{g}^{ab}, \tilde{A}_a)$ to the Einstein–Yang–Mills equations on $[0, \infty) \times \Sigma$ with regularity

$$(\tilde{g}^{ab}, \underline{\tilde{\nabla}}_d \tilde{g}^{ab}, \tilde{A}_a, \tilde{E}_a, \tilde{H}_{ab}) \in C^0\big([0, \infty), H^s(\Sigma)\big) \cap C^1\big([0, \infty), H^{s-1}(\Sigma)\big)$$

that satisfies the temporal and wave gauge constraints $\tilde{A}_a \tilde{T}^a = 0$ and $\tilde{Z}^a = 0$ on $[0, \infty) \times \Sigma$. Moreover, there exists a constant $C > 0$ such that for all $\tau \in [0, \infty)$,

$$\|\tilde{A}_a(\tau)\|_{H^s} + \|\tilde{E}_a(\tau)\|_{H^s} + \|\tilde{H}_{ab}(\tau)\|_{H^s} \leq C\sigma$$

$$\|\tilde{g}^{ab}(\tau) - \underline{\tilde{g}}^{ab}(\tau)\|_{H^s} + \|\underline{\tilde{\nabla}}_d \tilde{g}^{ab}(\tau)\|_{H^s} \leq C\left(\frac{\pi}{2} - \mathrm{gd}(H^{-1}\tau)\right)^2 \sigma \quad (\sim Ct^2\sigma).$$

Remarks on the Main Theorem

Main Theorem

Pic. (I) $\begin{cases} \text{Physical picture} \\ \text{wave gauge } \tilde{\mathcal{Z}}^a = 0 \\ \text{temporal gauge } \tilde{A}_a \tilde{T}^a = 0 \end{cases}$

Local existence of EYM is proper to state in this picture

Main Proof

$\begin{cases} \text{Conformal picture} \\ \text{wave gauge } \mathcal{Z}^a = 0 \\ \text{temporal gauge } A_a \nu^a = 0 \end{cases}$ Pic. (II)

Physical picture, wave gauge $\Longleftrightarrow$ Conformal picture, wave gauge

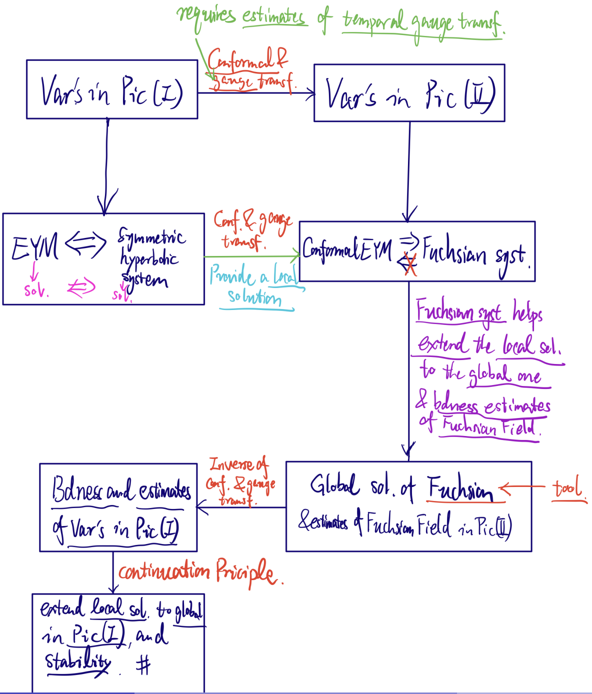
Temporal gauge (Pic. I) $\not\Longleftrightarrow$ temporal gauge (Pic. II)

Slightly different,

$\boxed{\begin{aligned} \tilde{T}^a &= (-\lambda)^{\frac{1}{2}} \tilde{g}^{ab} (d\tau)_a \\ \nu^a &= \underline{g}^{ab} (dt)_a \end{aligned}}$ After conformal transf. they do not coincide with each other

Long time behavior of EYM is more suitable in this picture

Since we have good Fuchsian Formulation ⟵ our main tool


requires estimates of temporal gauge transf.
Var's in Pic (I)
Conformal & gauge transf.
Var's in Pic (II)
EYM ⇔ Symmetric hyperbolic system
sol. ⇔ sol.
Conf. & gauge transf.
Provide a local solution
Conformal EYM ⇒ Fuchsian syst.
Fuchsian syst helps extend the local sol. to the global one
& bdness estimates of Fuchsian Field.
Global sol. of Fuchsian
& estimates of Fuchsian Field in Pic (II)
tool.
Inverse of conf. & gauge transf.
Bdness and estimates of Var's in Pic (I)
continuation Principle.
extend local sol. to global in Pic (I), and stability. #

Tool: Fuchsian formulations of Einstein–Yang–Mills system

$$
\begin{aligned}
B^\mu \partial_\mu u =& \frac{1}{t}\mathbf{B}\mathbf{P}u + G \quad && \text{in } [-1,0) \times \mathbb{T}^n, \\
u =& u_0 && \text{on } \{-1\} \times \mathbb{T}^n.
\end{aligned}
$$

Some main assumptions of this system

1. $\mathbf{P}$ is a constant, symmetric projection operator (Pick up decay variables by $\mathbf{P}$).
2. $\frac{1}{\gamma_1}\mathbb{I} \le \mathring{B}^0 \le \frac{1}{\kappa}\mathring{\mathbf{B}} \le \gamma_2 \mathbb{I}$ (Give right signs and determine dissipative effects).
3. $(B^\mu)^T = B^\mu, \quad [\mathbf{P}, \mathbf{B}] = \mathbf{P}\mathbf{B} - \mathbf{B}\mathbf{P} = 0.$
4. $\mathbf{P}^\perp B^0(t, \mathbf{P}^\perp u)\mathbf{P} = \mathbf{P} B^0(t, \mathbf{P}^\perp u)\mathbf{P}^\perp = 0.$
5. $|\mathbf{P}^\perp [D_u B^0(t,u)(B^0)^{-1}\mathbf{B}\mathbf{P}u]\mathbf{P}^\perp|_{op} \le \alpha|t| + \beta|\mathbf{P}u|^2.$
6. ...... e.g., G somehow allows $\sim 1/t$ (extra condition) and $\sim 1/\sqrt{t}$......

$(3,4,5$ gives how the variables coupling to each other$)$, and some regularity assumptions on the coefficients and remainders. Advantage: allow suitable coupling of variables.

The Global Existence Theorem of the Cauchy problem of Fuchsian equations

Theorem (Oliynyk, CMP, 2016)

Suppose that $k \geq \frac{n}{2} + 1$, $u_0 \in H^k(\mathbb{T}^n)$ and above conditions are fulfilled. Then there exists a $T_* \in (-1, 0)$, and a unique classical solution $u \in C^1([-1, T_*] \times \mathbb{T}^n)$ that satisfies $u \in C^0([-1, T_*], H^k) \cap C^1([-1, T_*], H^{k-1})$ and the energy estimate

$$\|u(t)\|_{H^k}^2 - \int_{-1}^{t} \frac{1}{\tau} \|\mathbf{P} u\|_{H^k}^2 d\tau \leq Ce^{C(t+1)}(\|u(-1)\|_{H^k}^2)$$

for all $-1 \leq t < T_*$, where $C = C(\|u\|_{L^\infty([-1,T_*),H^k)}, \gamma_1, \gamma_2, \kappa)$, and can be uniquely continued to a larger time interval $[T_0, T^*)$ for all $T^* \in (T_*, 0]$ provided $\|u\|_{L^\infty([-1,T_*),W^{1,\infty})} < \infty$.

This basic theorem has been generalized to more difficult cases and two parameter scales problems in the subsequent works by Oliynyk, L., Beyer, Olvera-Santamaría.

Intuitive toy model of Cauchy problem for Fuchsian system

- Rough idea: The following ODE dominated behaviors.
  Consider an ODE

$$\partial_t u = \frac{\beta}{t} u + (-t)^{-1+p} F(t), \quad \text{where} \quad 0 < p \leq 1, \beta > 0, t \in [-1, 0).$$

  Then

$$\partial_t \Big(u - \int_{-1}^{t} \frac{\beta}{s} u ds \Big) = (-t)^{-1+p} F(t).$$

  Integrating it yields

$$u - \int_{-1}^{t} \frac{\beta}{s} u ds \lesssim u_0 + 1 - (-t)^p.$$

  Further solving u leads to optimal decay estimates.
- The previous Theorem is obtained by adding conditions to make sure the Fuchsian system behaves like this toy model.

Fuchsian fields

1. Conformal transform

$$g_{ab} := e^{-2\Psi}\tilde{g}_{ab}, \quad F_{ab} := e^{-\Psi}\tilde{F}_{ab} \quad \text{and} \quad A_a := e^{-\frac{\Psi}{2}}\tilde{A}_a,$$

2. 3 + 1 Metric decomp. ($\nu_a = H(dt)_a$, $\nu^a = \underline{g}^{ab}\nu_b$ and $\underline{h}^a{}_b = \delta^a{}_b + \nu^a\nu_b$)

$$\lambda := g^{ab}\nu_a\nu_b, \quad \xi^c := g^{ab}\nu_a\underline{h}^c{}_b \quad \text{and} \quad h^{ab} := \underline{h}^a{}_c\underline{h}^b{}_d g^{cd},$$

3. Fuchsian gravitational field variables (rough)

$$m = \frac{1}{\breve{\mathtt{A}}t}(\lambda+1), \quad p^a = \frac{1}{\breve{\mathtt{B}}t}\xi^a, \quad m_d = \underline{\nabla}_d\lambda - \frac{1}{\breve{\mathtt{J}}Ht}(\lambda+1)\nu_d,$$

$$p^a{}_d = \underline{\nabla}_d\xi^a - \frac{1}{\breve{\mathtt{K}}Ht}\xi^a\nu_d, \quad s^{ab} = \mathfrak{h}^{ab} - \underline{h}^{ab},$$

$$s^{ab}{}_d = \underline{\nabla}_d(\mathfrak{h}^{ab} - \underline{h}^{ab}), \quad s = q, \quad s_d = \underline{\nabla}_d q,$$

$$\text{where} \quad q = \lambda + 1 + (3-n)\ln S \quad \text{and} \quad \mathfrak{h}^{ab} = \frac{1}{S}h^{ab}$$

4. 3 + 1 Fuchsian Yang-Mills field variables in a temporal gauge $A_a\nu^a = 0$

$$\bar{A}_b = A_a\underline{h}^a{}_b, \quad E_b = -\nu^p F_{pa}\underline{h}^a{}_b, \quad \mathcal{E}^a = -h^{ab}E_b, \quad \text{and} \quad H_{db} = \underline{h}^c{}_d F_{ca}\underline{h}^a{}_b.$$

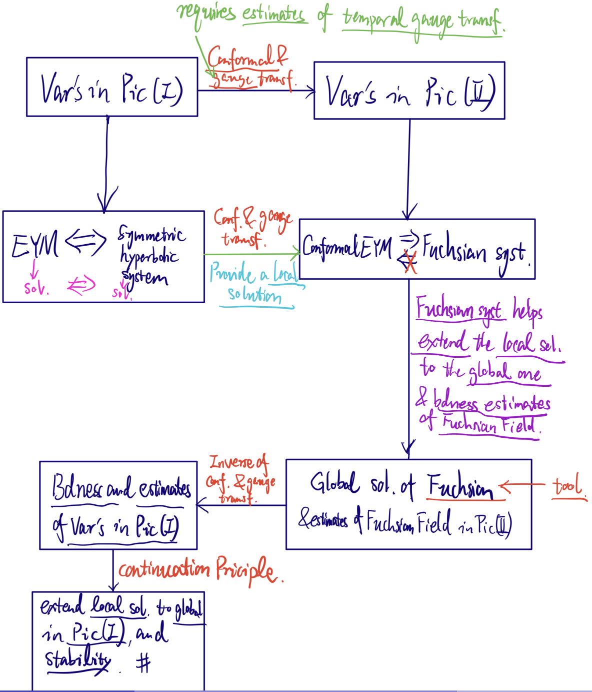
Remarks on the variables

- Suitable combinations kill higher order singular terms (getting the correct decay rate!)
- $1/\tau$ coeff. package more decays in the variables
- The to be determined constant coeff. $\breve{\mathtt{A}}$, $\breve{\mathtt{B}}$, $\breve{\mathtt{J}}$... are also crucial for killing bad terms, and the selections of them depend on the spacetime dimensions and they in fact helps to get correct decays.
- the conformal transform $F_{ab} := e^{-\Psi}\tilde{F}_{ab}$ and $A_a := e^{-\frac{\Psi}{2}}\tilde{A}_a$ are not optimal but it is easy for formulating to the Fuchsian system (not sharp in the Fuchsian, hence it is easy to verify the conditions). To amend this defect and to get the optimal decay, we improve the decay estimates by rescaling the conformal Yang–Mills fields,

$$\left(\mathring{\mathcal{E}}^e, \mathring{E}_d, \mathring{H}_{ab}, \mathring{A}_s\right)^{\mathrm{tr}} = \mathrm{diag}\{t^{-1}, t^{-1}, t^{-1}, t^{-\frac{1}{2}}\}\left(\mathcal{E}^e, E_d, H_{ab}, \bar{A}_s\right)^{\mathrm{tr}}.$$

Still a Fuchsian formulation and a posteriori estimate is obtained.

Fuchsian formulations of EYM system


requires estimates of temporal gauge transf.
Var's in Pic (I)
Conformal & gauge transf.
Var's in Pic (II)
EYM ⟺ Symmetric hyperbolic system
sol. ⟺ sol.
Conf. & gauge transf.
Provide a local solution
Conformal EYM ⇒ Fuchsian syst.
Fuchsian syst helps extend the local sol. to the global one & bdness estimates of Fuchsian Field.
Bdness and estimates of Var's in Pic (I)
Inverse of conf. & gauge transf.
Global sol. of Fuchsian &estimates of Fuchsian Field in Pic(II)
tool.
continuation Principle.
extend local sol. to global in Pic(I), and stability. #

Fuchsian expression of Einstein equations

Physical Einstein equation

$$\tilde{R}_{ab} - \frac{2}{n-2}\Lambda \tilde{g}_{ab} = \tilde{T}_{ab} - \frac{1}{n-2}\tilde{T}\tilde{g}_{ab}.$$

Conformal Einstein equation

$$R^{ab} = (n-2)(\nabla^a\nabla^b\Psi - \nabla^a\Psi\nabla^b\Psi) + \left(\Box\Psi + (n-2)\nabla^c\Psi\nabla_c\Psi + \frac{n-1}{H^2}e^{2\Psi}\right)g^{ab} + T^{ab} - \frac{1}{n-2}Tg^{ab},$$

where $T_{ab} = F_a{}^c F_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}$, $T^{ab} = g^{ac}g^{bd}T_{cd}$, $T = g_{cd}T^{cd}$, and we note that $T_{ab} = \tilde{T}_{ab}$ and $T = e^{2\Psi}\tilde{T}$.

Conformal wave gauge

we set

$$Z^a = X^a + Y^a$$

where

$$X^a := g^{bc} X^a{}_{bc} = -\underline{\nabla}_e g^{ae} + \frac{1}{2} g^{ae} g_{df} \underline{\nabla}_e g^{df},$$

$$Y^a = -(n-2)\nabla^a \Psi + \eta^a \quad \text{with} \quad \eta^a = (n-2)\underline{\nabla}^a \Psi = -\frac{n-2}{\tan(Ht)} \nu^a.$$

The conformal wave gauge that we employ for the conformal Einstein equations is then defined by the vanishing of the vector field, that is,

$$Z^a = 0.$$

The reduced Einstein equations are obtained from the Einstein equations by adding the term $-\nabla^{(a}Z^{b)} - \frac{1}{n-2}A^{ab}{}_{c}Z^{c}$ that vanishes when the wave gauge $Z^a = 0$ holds. The resulting equations are given by

$$R^{ab} - \nabla^{(a}Z^{b)} - \frac{1}{n-2}A^{ab}{}_{c}Z^{c} = T^{ab} - \frac{1}{n-2}g^{ab}g^{cd}T_{cd} + (n-2)(\nabla^a\nabla^b\Psi - \nabla^a\Psi\nabla^b\Psi)$$
$$+ \left(\Box\Psi + (n-2)\nabla^c\Psi\nabla_c\Psi + \frac{n-1}{H^2}e^{2\Psi}\right)g^{ab},$$

where $A^{ab}{}_{c}$ is defined by

$$A^{ab}{}_{c} = -X^{(a}\delta^{b)}{}_{c} + Y^{(a}\delta^{b)}{}_{c},$$

3 key choices of this transform

- Conformal factor: $\Psi = -\ln\left(\frac{\sin(Ht)}{H}\right)$ (partially include decay informations);
- Wave gauge: $Z^{\mu} = 0$ and the source function $Y^a := -(n-2)\nabla^a\Psi + (n-2)\underline{\nabla}^a\Psi$; (Desigened for killing redundant high order singular terms by costing nothing);
- Variables: Proper variables (Designed for killing higher order singular terms).

Eventually, the reduced conformal Einstein equations becomes

$$
\begin{aligned}
&\frac{1}{2}g^{cd}\underline{\nabla}_c\underline{\nabla}_d g^{ab} + \underline{R}^{ab} + P^{ab} + Q^{ab} + \frac{1}{n-2}X^a X^b - (n-2)\nu_c g^{c(a}\nu^{b)} \\
&+ \frac{n-2}{2\tan^2(Ht)}\nu^a(g^{bc} - \underline{g}^{bc})\nu_c + \frac{n-2}{2\tan^2(Ht)}\nu^b(g^{ac} - \underline{g}^{ac})\nu_c \\
=& \frac{n-2}{2\tan(Ht)}\nu^c\underline{\nabla}_c g^{ab} + \left(\frac{\lambda+1}{\sin^2(Ht)} + (n-2)\right) g^{ab} + g^{bd}F^{ac}F_{dc} \\
&- \frac{1}{2(n-2)}g^{ab}F^{cd}F_{cd}.
\end{aligned}
$$

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$$p^a{}_d = \underline{\nabla}_d\xi^a - \frac{1}{\breve{\mathtt{K}}Ht}\xi^a\nu_d, \quad s^{ab} = \mathfrak{h}^{ab} - \underline{h}^{ab},$$

$$s^{ab}{}_d = \underline{\nabla}_d(\mathfrak{h}^{ab} - \underline{h}^{ab}), \quad s = q, \quad s_d = \underline{\nabla}_d q,$$

$$\text{where} \quad q = \lambda + 1 + (3 - n)\ln S \quad \text{and} \quad \mathfrak{h}^{ab} = \frac{1}{S}h^{ab}$$

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$$\bar{A}_b = A_a\underline{h}^a{}_b, \quad E_b = -\nu^p F_{pa}\underline{h}^a{}_b, \quad \mathcal{E}^a = -h^{ab}E_b, \quad \text{and} \quad H_{db} = \underline{h}^c{}_d F_{ca}\underline{h}^a{}_b.$$

Decompose above eq into ν^c-direction and projections by h^c_d, we arrive at $\lambda+1$, ξ^a and $h^{ab}-\underline{h}^{ab}$ satisfy the wave equations

$$\frac{1}{2}g^{cd}\underline{\nabla}_c\underline{\nabla}_d(\lambda+1)+P^{ab}\nu_a\nu_b+Q^{ab}\nu_a\nu_b+\frac{1}{n-2}X^aX^b\nu_a\nu_b$$
$$=\frac{n-2}{2\tan(Ht)}\nu^c\underline{\nabla}_c(\lambda+1)+\frac{n-3}{\sin^2(Ht)}+\frac{(\lambda+1)^2}{\sin^2(Ht)}(\lambda+1)-(n-2)(\lambda+1)$$
$$+\left(\nu_a\nu_b g^{bd}g^{a\hat{a}}-\frac{1}{2(n-2)}\lambda g^{d\hat{a}}\right)g^{c\hat{c}}(H_{\hat{a}\hat{c}}-E_{\hat{a}}\nu_{\hat{c}}+\nu_{\hat{a}}E_{\hat{c}})(H_{dc}-E_d\nu_c+\nu_dE_c),$$

$$\frac{1}{2}g^{cd}\underline{\nabla}_c\underline{\nabla}_d\xi^e+P^{ab}\nu_a\underline{h}^e{}_b+Q^{ab}\nu_a\underline{h}^e{}_b+\frac{1}{n-2}X^aX^b\nu_a\underline{h}^e{}_b$$
$$=\frac{n-2}{2\tan(Ht)}\nu^c\underline{\nabla}_c\xi^e+\frac{n-2}{2\tan^2(Ht)}\xi^e+(\lambda+1)\xi^e\frac{1}{\sin^2(Ht)}+\frac{1}{2}(n-2)\xi^e$$
$$+\left(\nu_a\underline{h}^e{}_b g^{bd}g^{a\hat{a}}-\frac{1}{2(n-2)}\xi^e g^{d\hat{a}}\right)g^{c\hat{c}}(H_{\hat{a}\hat{c}}-E_{\hat{a}}\nu_{\hat{c}}+\nu_{\hat{a}}E_{\hat{c}})(H_{dc}-E_d\nu_c+\nu_dE_c),$$

and

$$\frac{1}{2}g^{cd}\underline{\nabla}_c\underline{\nabla}_d\left(h^{ef}-\underline{h}^{ef}\right)+\underline{h}^e{}_a\underline{h}^f{}_bP^{ab}+\underline{h}^e{}_a\underline{h}^f{}_bQ^{ab}+\frac{1}{n-2}\underline{h}^e{}_a\underline{h}^f{}_bX^aX^b$$
$$=\frac{n-2}{2\tan(Ht)}\nu^c\underline{\nabla}_c\left(h^{ef}-\underline{h}^{ef}\right)+h^{ef}\frac{\lambda+1}{\sin^2(Ht)}+(n-2)(\underline{h}^{ef}-h^{ef})$$
$$+\left(\underline{h}^e{}_a\underline{h}^f{}_b g^{bd}g^{a\hat{a}}-\frac{1}{2(n-2)}h^{ef}g^{d\hat{a}}\right)g^{c\hat{c}}(H_{\hat{a}\hat{c}}-E_{\hat{a}}\nu_{\hat{c}}+\nu_{\hat{a}}E_{\hat{c}})(H_{dc}-E_d\nu_c+\nu_dE_c),$$

Moreover q and $\mathfrak{h}^{ab} - \underline{h}^{ab}$ satisfy the wave equations

$$\frac{1}{2}g^{cd}\underline{\nabla}_c\underline{\nabla}_d q - \frac{3-n}{2(n-1)}g^{cd}\underline{\nabla}_c h_{ef}\underline{\nabla}_d h^{ef} + \frac{3-n}{n-1}h_{ab}P^{ab} + \frac{3-n}{n-1}h_{ab}Q^{ab}$$
$$+ \frac{3-n}{(n-1)(n-2)}h_{ab}X^aX^b + P^{ab}\nu_a\nu_b + Q^{ab}\nu_a\nu_b + \frac{1}{n-2}X^aX^b\nu_a\nu_b$$
$$= \frac{(n-2)}{2\tan(Ht)}\nu^c\underline{\nabla}_c q + (\lambda+1)^2\frac{1}{\sin^2(Ht)} - (n-2)(\lambda+1) + \Big(\frac{3-n}{n-1}h_{ab}g^{bd}g^{a\hat{a}}$$
$$- \frac{3-n+\lambda}{2(n-2)}g^{d\hat{a}} + \nu_a\nu_b g^{bd}g^{a\hat{a}}\Big)\cdot g^{c\hat{c}}(H_{\hat{a}\hat{c}} - E_{\hat{a}}\nu_{\hat{c}} + \nu_{\hat{a}}E_{\hat{c}})(H_{dc} - E_d\nu_c + \nu_d E_c),$$

and

$$\frac{1}{2}g^{cd}\underline{\nabla}_c\underline{\nabla}_d(\mathfrak{h}^{ab} - \underline{h}^{ab}) - \frac{1}{2}g^{cd}\underline{\nabla}_c(S^{-1}\mathcal{L}^{ab}{}_{ef})\underline{\nabla}_d h^{ef} + S^{-1}\mathcal{L}^{ab}{}_{ef}\,\underline{h}^e{}_a\underline{h}^f{}_b P^{ab}$$
$$+ S^{-1}\mathcal{L}^{ab}{}_{ef}\,\underline{h}^e{}_a\underline{h}^f{}_b Q^{ab} + \frac{1}{n-2}S^{-1}\mathcal{L}^{ab}{}_{ef}\,\underline{h}^e{}_a\underline{h}^f{}_b X^aX^b$$
$$= \frac{n-2}{2\tan(Ht)}\nu^c\underline{\nabla}_c(\mathfrak{h}^{ab} - \underline{h}^{ab}) - (n-2)S^{-1}\mathcal{L}^{ab}{}_{cd}(\mathfrak{h}^{cd} - \underline{h}^{cd})$$
$$+ S^{-1}\mathcal{L}^{ab}{}_{ef}\,\underline{h}^e{}_a\underline{h}^f{}_b g^{bd}g^{a\hat{a}}g^{c\hat{c}}(H_{\hat{a}\hat{c}} - E_{\hat{a}}\nu_{\hat{c}} + \nu_{\hat{a}}E_{\hat{c}})(H_{dc} - E_d\nu_c + \nu_d E_c),$$

respectively.

Remark on equations of these strange variables

Ideas: We use these strange variables since the remainders of these equations only include the $\sim 1/t$ singular terms (in order to obtain the target Fuchsian formulations) without higher order singular terms (otherwise, there are higher singular terms)! Suitable transformations of the original variables and linear combinations are applied (to kill the higher order singular terms like $\sim 1/t^2$).

We use repeatedly the symmetrizing tensor Q^{edc} defined by

$$Q^{edc} = \nu^e g^{dc} + \nu^d g^{ec} - \nu^c g^{ed}, \quad (i.e., Q^{edc} = Q^{dec})$$

to symmetrize the system. e.g.,

$$\begin{aligned}
\nu^e(g^{cd}\underline{\nabla}_c s^{ab}{}_d) &= Q^{edc}\underline{\nabla}_c s^{ab}{}_d - (\nu^d g^{ec}\underline{\nabla}_c s^{ab}{}_d - \nu^c g^{ed}\underline{\nabla}_c s^{ab}{}_d) \\
&= Q^{edc}\underline{\nabla}_c s^{ab}{}_d - \nu^d g^{ec}(\underline{\nabla}_c\underline{\nabla}_d s^{ab} - \underline{\nabla}_d\underline{\nabla}_c s^{ab}) \\
&= Q^{edc}\underline{\nabla}_c s^{ab}{}_d + \nu^d g^{ec}(\underline{R}_{cdf}{}^a s^{fb} + \underline{R}_{cdf}{}^b s^{fa}).
\end{aligned}$$

Remark: The advantage will be clear in Yang–Mills equations.

Along with Q^{edc}, we also use the time rescaled perturbation variables m, m_d, p^a_d, s^{ab}... (with parameter $\breve{\mathtt{A}}$,...) to rewrite above equations (write the 2nd order wave eqs. into the 1st order symmetric hyperbolic system and lengthy calculations...),

$$-\bar{\mathbf{A}}^0_1\nu^c\underline{\nabla}_c\begin{pmatrix}-\nu^e m_e\\ \underline{h}^e{}_{\hat{e}}m_e\\ m\end{pmatrix}+\bar{\mathbf{A}}^c_1\underline{h}^b{}_c\underline{\nabla}_b\begin{pmatrix}-\nu^e m_e\\ \underline{h}^e{}_{\hat{e}}m_e\\ m\end{pmatrix}=\frac{1}{Ht}\bar{\mathscr{B}}_1\begin{pmatrix}-\nu^e m_e\\ \underline{h}^e{}_{\hat{e}}m_e\\ m\end{pmatrix}+\bar{G}_1(t,\mathbf{U}),$$

$$-\bar{\mathbf{A}}^0_2\nu^c\underline{\nabla}_c\begin{pmatrix}-\nu^e p^a{}_e\\ \underline{h}^e{}_{\hat{e}}p^a{}_e\\ p^a\end{pmatrix}+\bar{\mathbf{A}}^c_2\underline{h}^b{}_c\underline{\nabla}_b\begin{pmatrix}-\nu^e p^a{}_e\\ \underline{h}^e{}_{\hat{e}}p^a{}_e\\ p^a\end{pmatrix}=\frac{1}{Ht}\bar{\mathscr{B}}_2\begin{pmatrix}-\nu^e p^a{}_e\\ \underline{h}^e{}_{\hat{e}}p^a{}_e\\ p^a\end{pmatrix}+\bar{G}_2(t,\mathbf{U}),$$

$$-\bar{\mathbf{A}}^0_3\nu^c\underline{\nabla}_c\begin{pmatrix}-\nu^e s^{\hat{a}\hat{b}}{}_e\\ \underline{h}^e{}_{\hat{e}}s^{\hat{a}\hat{b}}{}_e\\ s^{\hat{a}\hat{b}}\end{pmatrix}+\bar{\mathbf{A}}^c_3\underline{h}^b{}_c\underline{\nabla}_b\begin{pmatrix}-\nu^e s^{\hat{a}\hat{b}}{}_e\\ \underline{h}^e{}_{\hat{e}}s^{\hat{a}\hat{b}}{}_e\\ s^{\hat{a}\hat{b}}\end{pmatrix}=\frac{1}{Ht}\bar{\mathscr{B}}_3\begin{pmatrix}-\nu^e s^{\hat{a}\hat{b}}{}_e\\ \underline{h}^e{}_{\hat{e}}s^{\hat{a}\hat{b}}{}_e\\ s^{\hat{a}\hat{b}}\end{pmatrix}+\bar{G}_3(t,\mathbf{U}),$$

$$-\bar{\mathbf{A}}^0_4\nu^c\underline{\nabla}_c\begin{pmatrix}-\nu^e s_e\\ \underline{h}^e{}_{\hat{e}}s_e\\ s\end{pmatrix}+\bar{\mathbf{A}}^c_4\underline{h}^b{}_c\underline{\nabla}_b\begin{pmatrix}-\nu^e s_e\\ \underline{h}^e{}_{\hat{e}}s_e\\ s\end{pmatrix}=\frac{1}{Ht}\bar{\mathscr{B}}_4\begin{pmatrix}-\nu^e s_e\\ \underline{h}^e{}_{\hat{e}}s_e\\ s\end{pmatrix}+\bar{G}_4(t,\mathbf{U}),$$

where

$$\bar{\mathbf{A}}_1^0 = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & \underline{h}_{f\hat{c}} h^{f\hat{e}} & 0 \\ 0 & 0 & \breve{\mathtt{E}}(-\lambda) \end{pmatrix}, \quad \bar{\mathbf{A}}_1^c \underline{h}^b{}_c = \begin{pmatrix} -2\xi^b & -h^{\hat{e}b} & 0 \\ -\underline{h}_{f\hat{c}} h^{fb} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\mathscr{B}}_1 = \begin{pmatrix} \left(n-2-\frac{1}{\breve{\mathtt{J}}}\right)(-\lambda) & 0 & \left(2-\frac{1}{\breve{\mathtt{J}}}\right)\left(\frac{1}{\breve{\mathtt{J}}}-n+3\right)\frac{\breve{\mathtt{A}}}{H}(-\lambda) \\ 0 & \frac{1}{\breve{\mathtt{J}}}\underline{h}_{f\hat{c}} h^{f\hat{e}} & 0 \\ \breve{\mathtt{E}}\frac{H}{\breve{\mathtt{A}}}(-\lambda) & 0 & \breve{\mathtt{E}}\left(\frac{1}{\breve{\mathtt{J}}}-1\right)(-\lambda) \end{pmatrix},$$

$$\bar{G}_1(t,\mathbf{U}) = \begin{pmatrix} \nu_e \triangle_1^e(t,\mathbf{U}) \\ \underline{h}_{\hat{c}e} \triangle_1^e(t,\mathbf{U}) \\ 0 \end{pmatrix}, \qquad \bar{G}_2(t,\mathbf{U}) = \begin{pmatrix} \nu_e \triangle_2^{ea}(t,\mathbf{U}) \\ \underline{h}_{\hat{d}e} \triangle_2^{ea}(t,\mathbf{U}) \\ 0 \end{pmatrix},$$

$$\bar{\mathbf{A}}_3^0 = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & \underline{h}_{fa} h^{f\hat{e}} & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \qquad \bar{\mathbf{A}}_3^c \underline{h}^b{}_c = \begin{pmatrix} -2\xi^b & -h^{\hat{e}b} & 0 \\ -\underline{h}_{fa} h^{fb} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\mathscr{B}}_3 = \begin{pmatrix} -\lambda(n-2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \bar{\mathscr{B}}_4 = \begin{pmatrix} -\lambda(n-2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{G}_3(t,\mathbf{U}) = \begin{pmatrix} \nu_e \triangle_3^{e\hat{a}\hat{b}}(t,\mathbf{U}) \\ \underline{h}_{ae} \triangle_3^{e\hat{a}\hat{b}}(t,\mathbf{U}) \\ \lambda \nu^e s^{\hat{a}\hat{b}}{}_e \end{pmatrix}, \qquad \bar{G}_4(t,\mathbf{U}) = \begin{pmatrix} \nu_e \triangle_4^e(t,\mathbf{U}) \\ \underline{h}_{\hat{a}e} \triangle_4^e(t,\mathbf{U}) \\ \lambda \nu^e s_e \end{pmatrix}.$$

Fuchsian expression of Yang–Mills equations

The conformal Yang–Mills equations

$$\nabla^a F_{ab} = -(n-3)\nabla^a \Psi F_{ab} - e^{\frac{\Psi}{2}} g^{ac}[A_c, F_{ab}],$$
$$\nabla_{[a} F_{bc]} = -\nabla_{[a}\Psi \cdot F_{bc]} - e^{\frac{\Psi}{2}}[A_{[a}, F_{bc]}],$$

noting

$$\nabla_a F_{bc} = \underline{\nabla}_a F_{bc} - X^d_{ab} F_{dc} - X^d_{ac} F_{bd},$$

then

$$g^{ba}\underline{\nabla}_b F_{ac} = \frac{(n-3)}{\tan(Ht)} g^{ba}\nu_a F_{bc} + X^d F_{dc} + g^{ba} X^d_{bc} F_{ad} - \frac{\sqrt{H}}{\sqrt{\sin(Ht)}} g^{ab}[A_b, F_{ac}],$$
$$\underline{\nabla}_{[b} F_{ac]} = \frac{1}{\tan(Ht)} \nu_{[b} F_{ac]} - \frac{\sqrt{H}}{\sqrt{\sin(Ht)}} [A_{[b}, F_{ac]}].$$

Question: How to express above system into the Fuchsian form? Symmetrizing operator Q^{edc} like Einstein equations?

Answer: Yes! and No!

Symmetrization of the Yang–Mills system

Step 1: Q^{edc}-Symmetrization:

Multiplying the 1st YM equation (conservation) on both sides by the spatial projection tensor $\underline{h}^a{}_b$ gives

$$Q^{edc}\underline{\nabla}_c F_{da}\underline{h}^a{}_b + \overbrace{(-\nu^d g^{ec} + \nu^c g^{ed})\underline{\nabla}_c F_{da}\underline{h}^a{}_b}^{=\nu^d g^{ec}(\underline{\nabla}_c F_{ad}+\underline{\nabla}_d F_{ca})\underline{h}^a{}_b.\ \text{Using the 2nd YM eq. Bianchi eq. to replace}}$$
$$= \frac{(n-3)}{\tan(Ht)}\nu^e g^{dc}\nu_c F_{da}\underline{h}^a{}_b + \nu^e X^d F_{da}\underline{h}^a{}_b + \nu^e g^{\hat{a}\hat{b}} X^d{}_{\hat{a}a} F_{\hat{b}d}\underline{h}^a{}_b$$
$$- \frac{\sqrt{H}}{\sqrt{\sin(Ht)}}\nu^e g^{cd}[A_d, F_{ca}]\underline{h}^a{}_b.$$

Decomposing F_{ab} to E_f and H_{lf}, and using the relation between F_{ab} and A_a in the temporal gauge,

$$-\mathscr{A}^0\nu^c\underline{\nabla}_c\begin{pmatrix}E_{\hat{a}}\\H_{l\hat{a}}\\\bar{A}_{\hat{a}}\end{pmatrix} + \mathscr{A}^c_{\hat{a}}\underline{h}^{\hat{c}}{}_c\underline{\nabla}_{\hat{c}}\begin{pmatrix}E_f\\H_{lf}\\\bar{A}_f\end{pmatrix} = \frac{1}{Ht}\mathscr{B}\begin{pmatrix}E_{\hat{a}}\\H_{l\hat{a}}\\\bar{A}_{\hat{a}}\end{pmatrix} + \frac{1}{\sqrt{t}}\begin{pmatrix}\Xi_{1\hat{a}}\\\Xi^h_{2\hat{a}}\\\Xi_{3\hat{a}}\end{pmatrix} + \begin{pmatrix}\widehat{\Delta}_{1\hat{a}}\\\widehat{\Delta}^h_{2\hat{a}}\\\widehat{\Delta}_{3\hat{a}}\end{pmatrix},$$

Denoted, in this talk, by EHA-equation

$$-\mathscr{A}^0\nu^c\underline{\nabla}_c\begin{pmatrix}E_{\hat{a}}\\H_{l\hat{a}}\\\bar{A}_{\hat{a}}\end{pmatrix}+\mathscr{A}^c_{\hat{a}}\underline{h}^{\hat{c}}{}_c\underline{\nabla}_{\hat{c}}\begin{pmatrix}E_f\\H_{lf}\\\bar{A}_f\end{pmatrix}=\frac{1}{Ht}\mathscr{B}\begin{pmatrix}E_{\hat{a}}\\H_{l\hat{a}}\\\bar{A}_{\hat{a}}\end{pmatrix}+\frac{1}{\sqrt{t}}\begin{pmatrix}\Xi_{1\hat{a}}\\\Xi^h_{2\hat{a}}\\\Xi_{3\hat{a}}\end{pmatrix}+\begin{pmatrix}\widehat{\Delta}_{1\hat{a}}\\\widehat{\Delta}^h_{2\hat{a}}\\\widehat{\Delta}_{3\hat{a}}\end{pmatrix},$$

where

$$\mathscr{A}^0=\begin{pmatrix}-\lambda&0&0\\0&h^{hl}&0\\0&0&1\end{pmatrix},\quad\mathscr{B}=\begin{pmatrix}-(n-3)\lambda&-(n-4)\xi^l&0\\0&h^{hl}&0\\0&0&\frac{1}{2}\end{pmatrix},$$

$$\mathscr{A}^c_{\hat{a}}\underline{h}^{\hat{c}}{}_c=\begin{pmatrix}-2h^f{}_{\hat{a}}g^{dc}\nu_d\underline{h}^{\hat{c}}{}_c+\underline{h}^{\hat{c}}{}_{\hat{a}}g^{dc}\nu_d\underline{h}^f{}_c&-\underline{h}^f{}_{\hat{a}}h^{l\hat{c}}&0\\\overbrace{\underline{h}^{\hat{c}}{}_{\hat{a}}h^{hf}}^{\text{break the symmetry}}-\underline{h}^f{}_{\hat{a}}h^{h\hat{c}}&0&0\\0&0&0\end{pmatrix},$$

and there exists constants $\iota>0$ and $R>0$ such that the maps Ξ_i, $\widehat{\Delta}_i$, $i=1,2,3$, are analytic for $(t,\mathbf{U})\in(-\iota,\frac{\pi}{H})\times B_R(0)$.

Symmetrization of the Yang–Mills system

Step 2: New variable $\mathcal{E}^a$-Symmetrization: To remedy the defect from the red term, supplement with an additional equation by introducing a new variable $\mathcal{E}^a$ to symmetrize it (ideas from C.L. and J. Wang: New formulations of EYM system and local existence. But significant difference (later)).

Introduce a new variable

$$\mathcal{E}^a = -g^{e\hat{b}}\underline{h}^a{}_e E_{\hat{b}},$$

note that the E_b and $\mathcal{E}^a$ are related by

$$\mathcal{E}^a = -h^{a\hat{b}} E_{\hat{b}} \quad \text{and} \quad E_b = -\mathcal{E}^a g_{ab} - g^{e\hat{b}} \nu_e E_{\hat{b}} \nu^a g_{ab}.$$

and supplement an additional equation ($\nu^c \times$ the first YM equation)

$$g^{ba}\underline{\nabla}_b E_a = \frac{\breve{\mathtt{B}}(n-3)t}{\tan(Ht)} p^b E_b + (X^d E_d + \nu^c g^{ba} X^d{}_{bc} F_{ad}) - \frac{\sqrt{H}}{\sqrt{\sin(Ht)}} h^{ab}[\bar{A}_b, E_a],$$

Mention the following equation, in this talk, EEHA equation

$$-\check{\mathbf{A}}^0\nu^c\underline{\nabla}_c\begin{pmatrix}\mathcal{E}^e\\ E_{\hat{d}}\\ H_{\hat{a}\hat{b}}\\ \bar{A}_s\end{pmatrix}-\check{\mathbf{A}}^f\underline{h}^c{}_f\underline{\nabla}_c\begin{pmatrix}\mathcal{E}^e\\ E_{\hat{d}}\\ H_{\hat{a}\hat{b}}\\ \bar{A}_s\end{pmatrix}=\frac{1}{Ht}\check{\mathscr{B}}\begin{pmatrix}\mathcal{E}^e\\ E_{\hat{d}}\\ H_{\hat{a}\hat{b}}\\ \bar{A}_s\end{pmatrix}+\frac{1}{\sqrt{t}}\begin{pmatrix}-\Xi_{1\hat{e}}\\ h^{d\hat{a}}\Xi_{1\hat{a}}\\ -h^{a\hat{a}}\Xi^{b}_{2\hat{a}}\\ h^{ra}\Xi_{3a}\end{pmatrix}+\begin{pmatrix}\mathfrak{D}^{\sharp}_{1\hat{e}}\\ \mathfrak{D}^{\sharp d}_{2}\\ \mathfrak{D}^{\sharp ab}_{3}\\ \mathfrak{D}^{\sharp r}_{4}\end{pmatrix},$$

where

$$\check{\mathbf{A}}^0=\begin{pmatrix}-\lambda\underline{h}^a{}_{\hat{e}}g_{ba}\underline{h}^b{}_e & -\lambda\nu^r g_{rs}\underline{h}^s{}_{\hat{e}}\xi^{\hat{d}} & 0 & 0\\ -\lambda\nu^r g_{rs}\underline{h}^s{}_e\xi^d & [-\lambda h^{\hat{d}d}-\lambda\nu^r\nu^s g_{rs}\xi^d\xi^{\hat{d}}+2\xi^d\xi^{\hat{d}}] & 0 & 0\\ 0 & 0 & h^{\hat{a}a}h^{\hat{b}b} & 0\\ 0 & 0 & 0 & h^{rs}\end{pmatrix},$$

$$\check{\mathbf{A}}^f\underline{h}^c{}_f=\begin{pmatrix}2\xi^c\underline{h}^a{}_{\hat{e}}g_{ba}\underline{h}^b{}_e & [2\xi^c\nu^r g_{rs}\underline{h}^s{}_{\hat{e}}+\underline{h}^c{}_{\hat{e}}]\xi^{\hat{d}} & -h^{\hat{a}c}\underline{h}^{\hat{b}}{}_{\hat{e}} & 0\\ [2\xi^c\nu^r g_{rs}\underline{h}^s{}_e+\underline{h}^c{}_e]\xi^d & [2\nu^r\nu^s g_{rs}\xi^c\xi^{\hat{d}}\xi^d-2\xi^{(d}h^{\hat{d})c}+2\xi^c h^{\hat{d}d}] & -h^{\hat{a}d}h^{\hat{b}c} & 0\\ -h^{ac}\underline{h}^b{}_e & -h^{a\hat{d}}h^{bc} & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\check{\mathscr{B}}=\begin{pmatrix}-(n-3)\lambda\underline{h}^a{}_{\hat{e}}g_{ba}\underline{h}^b{}_e & 0 & 0 & 0\\ 0 & -(n-3)\lambda h^{\hat{d}d} & 0 & 0\\ 0 & 0 & h^{\hat{a}a}h^{\hat{b}b} & 0\\ 0 & 0 & 0 & \frac{1}{2}h^{rs}\end{pmatrix},$$

and there exist constants $\iota>0$ and $R>0$ such that the maps $\mathfrak{D}^{\sharp}_{1\hat{e}}(t,\mathbf{U})$, $\mathfrak{D}^{\sharp d}_{2}(t,\mathbf{U})$, $\mathfrak{D}^{\sharp ab}_{3}(t,\mathbf{U})$ and $\mathfrak{D}^{\sharp r}_{4}(t,\mathbf{U})$ are analytic for $(t,\mathbf{U})\in(-\iota,\frac{\pi}{H})\times B_R(0)$ and vanish for $\mathbf{U}=0$.

$$-\mathcal{A}^0 \nu^c \underline{\nabla}_c \begin{pmatrix} E_{\hat a} \\ H_{l\hat a} \\ \bar{A}_{\hat a} \end{pmatrix} + \mathcal{A}^c_{\hat a} \underline{h}^{\hat c}{}_c \underline{\nabla}_{\hat c} \begin{pmatrix} E_f \\ H_{lf} \\ \bar{A}_f \end{pmatrix} = \frac{1}{Ht}\mathcal{B} \begin{pmatrix} E_{\hat a} \\ H_{l\hat a} \\ \bar{A}_{\hat a} \end{pmatrix} + \frac{1}{\sqrt{t}} \begin{pmatrix} \Xi_{1\hat a} \\ \Xi^h_{2\hat a} \\ \Xi_{3\hat a} \end{pmatrix} + \begin{pmatrix} \widehat{\Delta}_{1\hat a} \\ \widehat{\Delta}^h_{2\hat a} \\ \widehat{\Delta}_{3\hat a} \end{pmatrix},$$

+additional eq.

$$-\acute{\mathbf{A}}^0 \nu^c \underline{\nabla}_c \begin{pmatrix} \mathcal{E}^e \\ E_{\hat d} \\ H_{\hat a \hat b} \\ \bar{A}_s \end{pmatrix} + \acute{\mathbf{A}}^f \underline{h}^c{}_f \underline{\nabla}_c \begin{pmatrix} \mathcal{E}^e \\ E_{\hat d} \\ H_{\hat a \hat b} \\ \bar{A}_s \end{pmatrix} = \frac{1}{Ht}\acute{\mathcal{B}} \begin{pmatrix} \mathcal{E}^e \\ E_{\hat d} \\ H_{\hat a \hat b} \\ \bar{A}_s \end{pmatrix} + \acute{G}_0(t, \mathbf{U}) + \frac{1}{\sqrt{t}} \acute{G}_1(t, \mathbf{U}),$$

Important remark

conformal Yang–Mills equations

$$\nabla^a F_{ab} = -(n-3)\nabla^a \Psi F_{ab} - e^{\frac{\Psi}{2}} g^{ac}[A_c, F_{ab}],$$
$$\nabla_{[a} F_{bc]} = -\nabla_{[a}\Psi \cdot F_{bc]} - e^{\frac{\Psi}{2}}[A_{[a}, F_{bc]}],$$

$$\Downarrow \qquad \cancel{\Uparrow}$$

$$-\acute{\mathbf{A}}^0 \nu^c \underline{\nabla}_c \begin{pmatrix} \mathcal{E}^e \\ E_{\hat d} \\ H_{\hat a \hat b} \\ \bar A_s \end{pmatrix} + \acute{\mathbf{A}}^f \underline{h}^c{}_f \underline{\nabla}_c \begin{pmatrix} \mathcal{E}^e \\ E_{\hat d} \\ H_{\hat a \hat b} \\ \bar A_s \end{pmatrix} = \frac{1}{Ht} \acute{\mathscr{B}} \begin{pmatrix} \mathcal{E}^e \\ E_{\hat d} \\ H_{\hat a \hat b} \\ \bar A_s \end{pmatrix} + \acute{G}_0(t, \mathbf{U}) + \frac{1}{\sqrt{t}} \acute{G}_1(t, \mathbf{U}),$$

EEHA equation is not equivalent to the conformal Yang–Mills equations expressed in the temporal gauge $A_d \nu^d = 0$. This is because the relation $\mathcal{E}^e = -h^{ea} E_a$ cannot be recovered from a solution $(\mathcal{E}^e, E_d, H_{ab}, \bar A_s)$ even in $\mathcal{E}^e = -h^{ea} E_a$ holds initially. Consequently, we cannot guarantee that the Yang–Mills equation will hold for a solution. However, it is true that if $(E_a, \bar A_b)$ solves the conformal Yang–Mills system in the temporal gauge, then the quadruple $(\mathcal{E}^e, E_d, H_{pq}, \bar A_s)$, where $\mathcal{E}^e = -h^{ea} E_a$ and H_{pq} is given in terms of $\bar A_a$, will solve EEHA eq. .

Complete reduced Einstein–Yang–Mills system

Let

$$\widehat{\mathbf{U}} = (-\nu^e m_e, \underline{h}^e{}_{\hat{e}} m_e, m, -\nu^e p^a{}_e, \underline{h}^e{}_{\hat{e}} p^a{}_e, p^a, -\nu^e s^{\hat{a}\hat{b}}{}_e, \underline{h}^e{}_{\hat{e}} s^{\hat{a}\hat{b}}{}_e, s^{\hat{a}\hat{b}}, -\nu^e s_e,$$
$$\underline{h}^e{}_{\hat{e}} s_e, s, \mathscr{E}^e, E_d, H_{\hat{a}\hat{b}}, \bar{A}_s)^{\mathrm{tr}}.$$

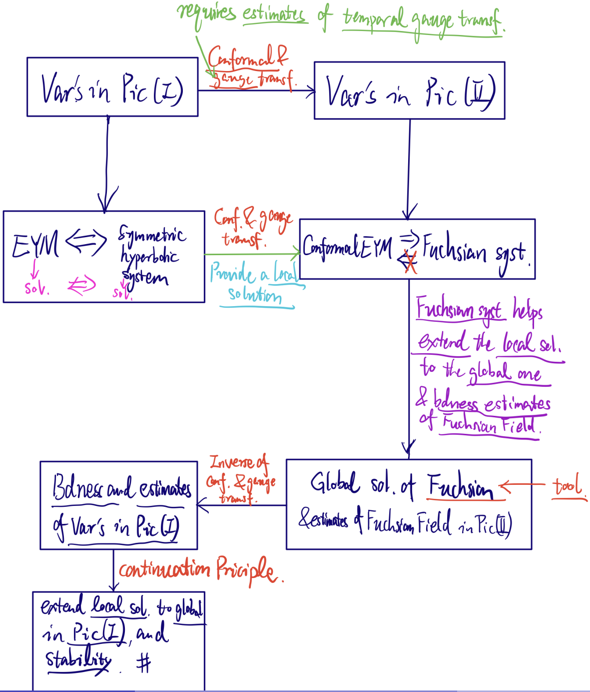
to get the complete non-degenerated singular symmetric hyperbolic system

$$-\widehat{\mathbf{A}}^0 \nu^c \underline{\nabla}_c \widehat{\mathbf{U}} + \widehat{\mathbf{A}}^b \underline{h}^c{}_b \underline{\nabla}_c \widehat{\mathbf{U}} = \frac{1}{Ht} \widehat{\mathscr{B}} \widehat{\mathbf{U}} + \widehat{G}(t, \widehat{\mathbf{U}}),$$

Verifications of Fuchsian system:

By choosing parameters $\breve{\mathtt{A}}, \breve{\mathtt{B}}, \breve{\mathtt{J}}, \breve{\mathtt{K}}, \breve{\mathtt{E}}, \breve{\mathtt{F}}$ (crucial with some tricks, and this selection reflects the decay rates of EYM fields evolving with time) correctly for different dimensions $n \geq 4$, above system becomes the Fuchsian equation (long verifications......OMIT, see our paper for detailed verifications). Then the global existence theorem of the Cauchy problem of Fuchsian equations conclude the Main Theorem for EYM system.

A sketch of the proof of Main Theorem


requires estimates of temporal gauge transf.
Conformal & gauge transf.
Var's in Pic (I)
Var's in Pic (II)
EYM ⇔ Symmetric hyperbolic system
sol. ⇔ sol.
Conf. & gauge transf.
Provide a local solution
Conformal EYM ⇒ Fuchsian syst.
Fuchsian syst helps extend the local sol. to the global one & bdness estimates of Fuchsian Field.
Global sol. of Fuchsian & estimates of Fuchsian Field in Pic(II)
tool.
Inverse of conf. & gauge transf.
Bdness and estimates of Var's in Pic(I)
continuation Principle.
extend local sol. to global in Pic(I), and stability. #

A sketch of the proof of Main Theorem

Step 1: A local solution of the Einstein–Yang–Mills equations.

- Y. Choquet-Bruhat: wave gauge + Lorentz gauge (hyperbolic form of EYM). She pointed out "the global results may be quite different".
- C. L. and J. Wang (2021), our companion paper: a new symmetric hyperbolic formulation of EYM. Fit for this global stability for EYM. (Use this result!)

Basic ideas of our companion local paper:

- The similar symmetrization (Q^{edc} symmetrization and $\mathcal{E}^a$ symmetrization) has been applied.
- We transform the EYM equations to the symmetric hyperbolic system and using the standard theory of symmetric hyperbolic system to conclude the local existence.

However, things are not so perfect......

Remarks on the local existence

<u>Main Difficulties and Methods:</u>

Question: Under the similar symmetrization (Q^{edc} symmetrization and $\mathcal{E}^a$ symmetrization), The EYM equations can not be "equivalent to" the symmetric hyperbolic system (or Fuchsian in this talk, only one direction is correct). Since during solving the symmetric hyperbolic system (or Fuchsian), we removed the constraint $\mathcal{E}^a := -g^{e\hat{b}}\underline{h}^a{}_e E_{\hat{b}} = -h^{a\hat{b}} E_{\hat{b}}$ (this constraint may not hold during the evolutions), the def. of the new variable $\mathcal{E}^a$. Therefore, $S_{EYM} \subset S_F$, where S_{EYM} denotes the solution set of the EYM system and the solution set of the symmetric hyperbolic system. This implies although we can obtain a unique solution to the symmetric system, it does not means this unique solution is a solution to the EYM system.

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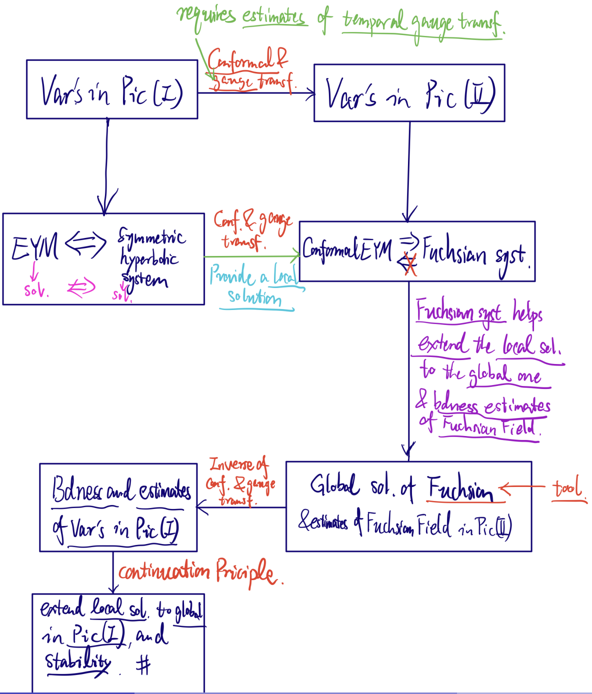
Answer: To solve this question, very roughly speaking, key tricks:

- We decompose all the similar YM-variables ($3+1$ decompositions) by $T^a := (-\lambda)^{-\frac{1}{2}} g^{ab}(dt)_b$ instead of $\nu^a := \underline{g}^{ab}(dt)_b$;
- The temporal gauge is chosen by $A_a T^a = 0$ instead of $A_a \nu^a = 0$;
- Under this variables and temporal gauge, we can prove the constraint $\mathcal{E}^a = -h^{a\hat{b}} E_{\hat{b}}$ (with the other YM constraints) hold during the evolutions for the symmetric hyperbolic system if it holds initially. Therefore, the EYM equations are "equivalent to" the symmetric hyperbolic system. Furthermore, the unique solution to sym. hyp.syst. is the unique solution to the EYM equations.

Remarks on the Step 1

- Using the above local existence of EYM equations and continuation principle, if we want to extend the solution, we have to control the solution bounded at the local interval.
- By transforming this local solution to the conformal picture with a slightly different temporal gauge (roughly speaking, $A_a T^a = 0$ becomes $A_a \nu^a = 0$), we are able to use above Fuchsian formulation (with the 2nd temporal gauge and conformal picture) to achieve this boundness and further use the continuation principle to extend the solution to global.
- This involves a temporal gauge transformation, in order to connect the estimates of variables in two temporal gauges, we have to investigate the gauge transform and estimate the transforms. i.e. we have to estimate the group $\tilde{\mathfrak{u}}$ (§6 in our paper, need some work) which is given by gauge transform

$$\tilde{A}_a = \tilde{\mathfrak{u}}^{-1} \tilde{A}^{\star}_a \tilde{\mathfrak{u}} + \tilde{\mathfrak{u}}^{-1} (d\tilde{\mathfrak{u}})_a \quad (i.e., \tilde{F}_{ab} = \tilde{\mathfrak{u}}^{-1} \tilde{F}^{\star}_{ab} \tilde{\mathfrak{u}}).$$


requires estimates of temporal gauge transf.
Conformal & gauge transf.
Var's in Pic (I)
Var's in Pic (II)
EYM ⟺ Symmetric hyperbolic system
sol. ⟺ sol.
Conf. & gauge transf.
Provide a local solution
Conformal EYM ⇒ Fuchsian syst.
Fuchsian syst helps extend the local sol. to the global one & bdness estimates of Fuchsian Field.
Global sol. of Fuchsian & estimates of Fuchsian Field in Pic(II)
tool.
Inverse of conf. & gauge transf.
Bdness and estimates of Var's in Pic (I)
continuation Principle.
extend local sol. to global in Pic(I), and stability. #

Step 2 and 3

Step 2: A local solution of the reduced conformal EYM system.

- After the conformal and the slightly temporal gauge transforms, we transform the previous local solution (in Step 1) to a local one of the reduced conformal EYM system under target wave and temporal gauge in this article, which prepares the solution to the Fuchsian system.

Step 3: A local solution of the Fuchsian system.

- By the global existence theorem and estimates of solution of Fuchsian system, we conclude the Fuchsian system has a unique global solution. Since EYM system "implies" the Fuchsian system (the contrary may not correct), the solution in Step 2 must be the unique solution of the Fuchsian system.

Step 4 and 5

Step 4: Fuchsian initial data bounds.

- In order to use the Fuchsian system to estimate the solution, we have to know the initial data of the Fuchsian system which comes from the small data in Step 1. Using conformal and gauge transforms to transform the data.

Step 5: Global existence and stability.

- After knowing the Fuchsian data small, we can use the global existence thereom of Fuchsian system to conclude the boundness of the Fuchsian variables which, by conformal and gauge transforms again, implies the boundness of the local solution in Step 1. Furthermore, back to Step 1, we achieve the aim of Step 1 and continuation principle conclude the Main Theorem. Complete the proof.

Thank you
for your attention!