On the nonlinear gravitational instabilities for Newtonian universes

Chao Liu

Email: chao_liu@hust.edu.cn

Huazhong University of Science and Technology

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Backgrounds

Classical Jeans instability (Static)

$$\begin{cases} \partial_t \rho + \partial_i(\rho v^i) = 0 \\ \partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0 \\ \delta^{ij}\partial_i\partial_j \phi = 4\pi G \rho \end{cases}$$

Euler – Poisson system

let $\rho = \underset{\substack{\uparrow \\ \text{constant.}}}{\underline{\rho_0}} + \tilde{\rho}$, $v^i = \underset{\substack{\uparrow \\ 0}}{\underline{v_0}} + \tilde{v}$, $\phi = \underset{\substack{\uparrow \\ \nabla\phi_0 = 0}}{\underline{\phi_0}} + \tilde{\phi}$, $\tilde{p} = c_s^2 \tilde{\rho}$

Contradicts Poisson.

① $\boxed{\text{linearization}} \Longrightarrow$

$$\begin{cases} \partial_t \tilde{\rho} + \rho_0 \partial_i \tilde{v}^i = 0 \\ \partial_t \tilde{v}^i + \frac{c_s^2}{\rho_0} \partial^i \tilde{\rho} + \partial^i \tilde{\phi} = 0 \\ \delta^{ij}\partial_i\partial_j \tilde{\phi} = 4\pi G \tilde{\rho} \end{cases} \xrightarrow{② \, \partial_i} \boxed{\partial_t^2 \tilde{\rho} - \underset{\substack{\uparrow \\ \text{Const.}}}{\underline{c_s^2}} \delta^{ij}\partial_i\partial_j \tilde{\rho} - \underset{\substack{\uparrow \\ \text{Const.}}}{\underline{4\pi G \rho_0}} \tilde{\rho} = 0}$$

$\Box \tilde{\rho} - k\tilde{\rho} = 0. \ (k > 0)$

$$\boxed{\partial_t^2\tilde{\rho} - c_s^2\delta^{ij}\partial_i\partial_j\tilde{\rho} - 4\pi G\rho_0\tilde{\rho} = 0}$$

$$\Big\Downarrow \text{ Fourier transform.}$$

$$\tilde{\rho}_k'' + (k^2c_s^2 - 4\pi G\rho_0)\tilde{\rho}_k = 0 \Rightarrow \begin{cases} \tilde{\rho}_k \propto \exp(\pm i\omega(k)t) \\ \omega(k) = \sqrt{k^2c_s^2 - 4\pi G\rho_0} \end{cases}$$

if $k^2c_s^2 - 4\pi G\rho_0 < 0.$ $\Leftrightarrow$ $k < \frac{\sqrt{4\pi G\rho_0}}{c_s} =: k_J$

$$\Rightarrow \boxed{\tilde{\rho}_k \propto \exp(\pm|\omega|t)}$$

exponentially growth.

Classical Jeans instability (expansion)

Expanding Newtonian Universe

$$\begin{cases} \rho = \rho_0(t) = \frac{1}{6\pi G t^2}, \quad v^i = v_0^i = \frac{2}{3t} x^i \quad \left(H(t) = \frac{2}{3t} \ \text{Hubble's law} \right) \\ \phi = \phi_0 = \frac{2}{3}\pi G \rho_0 |x|^2 \end{cases}$$

let $\rho = \rho_0 + \tilde{\rho}$, $v^i = v_0^i + \tilde{v}^i$, $\phi = \phi_0 + \tilde{\phi}$

$$\boxed{H = \frac{\dot{a}}{a}}$$

① Lagrangian coord. (comoving with Hubble flow) : $x^i = a(t)\, q^i$

density contrast

② Linearization Euler – Poisson

③ ∂_i (momentum conservation) ← Poisson eq. ← continuity eq.

let $e = \frac{\tilde{\rho}}{\rho_0}$ ⟹

$$\boxed{\partial_t^2 e + 2\underbrace{H}_{=\frac{4}{3t}}\dot{e} - \frac{c_s^2}{a^2}\delta^{ij}\partial_i\partial_j e - 4\pi G \rho_0 e = 0}$$

$$\boxed{\partial_t^2 e + \frac{4}{3t}\dot{e} - \frac{c_s^2}{a^2}\delta^{ij}\partial_i\partial_j e - \frac{2}{3t^2}e = 0}$$

$\Downarrow$ Fourier transform

$$e_k'' + \frac{4}{3t}e_k' + \left(\frac{c_s^2 k^2}{a^2} - \frac{2}{3t^2}\right)e_k = 0.$$

$\Downarrow$ c_s small (pressure small)

$$e_k'' + \frac{4}{3t}e_k' - \frac{2}{3t^2}e_k = 0$$

$\Downarrow$ Euler ODE

$$e_k = C_1 t^{-1} + C_2 t^{\frac{2}{3}} \Rightarrow |e| \sim t^{\frac{2}{3}}.$$

Slightly nonlinear Jeans instability

Question:

$$\ddot{\varrho} + \frac{4}{3t}\dot{\varrho} - \tilde{\kappa} t^{-2\gamma+\frac{2}{3}}\Delta\varrho - \frac{2}{3t^2}\varrho = (\gamma-1)\tilde{\kappa} t^{-2\gamma+\frac{2}{3}}\frac{D^i\varrho D_i\varrho}{1+\varrho}. \tag{1}$$

Theorem

Suppose $s \in \mathbb{Z}_{\geq 3}$ and $\gamma > 1$ are constants and $\mathring{\varrho} := \varrho|_{t=1}$ and $\mathring{\varrho}_\mu := (\partial_\mu \varrho)|_{t=1}$ $(\mu = 0, \cdots, 3)$. Let the initial data of the density satisfies an estimate

$$\left\| \mathring{\varrho} - \frac{\beta}{2} \right\|_{H^s(\mathbb{T}^3)} + \left\| \mathring{\varrho}_0 - \frac{\beta}{3} \right\|_{H^s(\mathbb{T}^3)} + \|\mathring{\varrho}_i\|_{H^s(\mathbb{T}^3)} \leq \beta_0, \tag{2}$$

where $0 < \beta < +\infty$ is any given constant and $\beta_0 > 0$ is a small enough constant. Then the solution of equation (1) satisfies

$$\frac{1}{4}\beta t^{\frac{2}{3}} \leq \varrho \leq \frac{3}{4}\beta t^{\frac{2}{3}}$$

for every $(t, \boldsymbol{q})$.

Methods

- Non-Fourier based method: Fuchsian formulations (energy method)
- Main difficulties:
  1. Find a compactified time $\tau \in [-1, 0)$ for physical time $t \in [t_0, \infty)$ such that $\tau = g(t)$.
  2. Select proper Fuchsian fields (similar define suitable energies)

Tool: Fuchsian formulations

$$\begin{aligned} B^\mu \partial_\mu u =& \frac{1}{t}\mathbf{B}\mathbf{P}u + G \quad \text{in } [-1,0)\times \mathbb{T}^n, \\ u =& u_0 \quad\quad\quad\quad\quad \text{on } \{-1\}\times\mathbb{T}^n. \end{aligned}$$

Some main assumptions of this system

1. $\mathbf{P}$ is a constant, symmetric projection operator (Pick up decay variables by $\mathbf{P}$).
2. $\frac{1}{\gamma_1}\mathbb{I} \leq \mathring{B}^0 \leq \frac{1}{\kappa}\mathring{\mathbf{B}} \leq \gamma_2\mathbb{I}$ (Give right signs and determine dissipative effects).
3. $(B^\mu)^T = B^\mu, \quad [\mathbf{P},\mathbf{B}] = \mathbf{PB} - \mathbf{BP} = 0.$
4. $\mathbf{P}^\perp B^0(t,\mathbf{P}^\perp u)\mathbf{P} = \mathbf{P}B^0(t,\mathbf{P}^\perp u)\mathbf{P}^\perp = 0.$
5. $|\mathbf{P}^\perp[D_u B^0(t,u)(B^0)^{-1}\mathbf{BP}u]\mathbf{P}^\perp|_{op} \leq \alpha|t| + \beta|\mathbf{P}u|^2.$
6. ...... e.g., B^i, G somehow allows $\sim 1/t$ (extra condition) and $\sim 1/\sqrt{t}$......

$(3,4,5$ gives how the variables coupling to each other), and some regularity assumptions on the coefficients and remainders. Advantage: allow suitable coupling of variables.

The Global Existence Theorem of the Cauchy problem of Fuchsian equations

Theorem (Oliynyk, 2016)

Suppose that $k \geq \frac{n}{2} + 1$, $u_0 \in H^k(\mathbb{T}^n)$ and above conditions are fulfilled. Then there exists a $T_* \in (-1, 0)$, and a unique classical solution $u \in C^1([-1, T_*] \times \mathbb{T}^n)$ that satisfies $u \in C^0([-1, T_*], H^k) \cap C^1([-1, T_*], H^{k-1})$ and the energy estimate

$$\|u(t)\|_{H^k}^2 - \int_{-1}^{t} \frac{1}{\tau} \|\mathbf{P}u\|_{H^k}^2 d\tau \leq Ce^{C(t+1)}(\|u(-1)\|_{H^k}^2)$$

for all $-1 \leq t < T_*$, where $C = C(\|u\|_{L^\infty([-1,T_*),H^k)}, \gamma_1, \gamma_2, \kappa)$, and can be uniquely continued to a larger time interval $[T_0, T^*)$ for all $T^* \in (T_*, 0]$ provided $\|u\|_{L^\infty([-1,T_*),W^{1,\infty})} < \infty$.

This basic theorem has been generalized to more difficult cases and two parameter scales problems in the subsequent works by Oliynyk, L., Beyer, Olvera-Santamaría.

Intuitive toy model of Cauchy problem for Fuchsian system

- Rough idea: The following ODE dominated behaviors.
  Consider an ODE

  $$\partial_t u = \frac{\beta}{t} u + (-t)^{-1+p} F(t), \quad \text{where} \quad 0 < p \leq 1, \beta > 0, t \in [-1, 0).$$

  Then

  $$\partial_t \Big(u - \int_{-1}^{t} \frac{\beta}{s} u ds \Big) = (-t)^{-1+p} F(t).$$

  Integrating it yields

  $$u - \int_{-1}^{t} \frac{\beta}{s} u ds \lesssim u_0 + 1 - (-t)^p.$$

  Further solving u leads to optimal decay estimates.
- The previous Theorem is obtained by adding conditions to make sure the Fuchsian system behaves like this toy model.

Compactified time:

$$\tau = \frac{1}{t} \in (0, 1]$$

Fuchsian fields

$$\begin{cases} u(\tau, \boldsymbol{q}) := \dfrac{\sqrt{6}}{3} t^{-\frac{2}{3}} \varrho(t, \boldsymbol{q}) - \dfrac{\sqrt{6}}{6} \beta, \\ u_0(\tau, \boldsymbol{q}) := t^{\frac{1}{3}} \partial_t \varrho(t, \boldsymbol{q}) - \dfrac{1}{3} \beta, \\ u_i(\tau, \boldsymbol{q}) := t^{\frac{2}{3} - \gamma} \partial_i \varrho(t, \boldsymbol{q}). \end{cases}$$

Fuchsian formulations

$$\ddot{\varrho}+\frac{4}{3t}\dot{\varrho}-\tilde{\kappa}t^{-2\gamma+\frac{2}{3}}\Delta\varrho-\frac{2}{3t^2}\varrho=(\gamma-1)\tilde{\kappa}t^{-2\gamma+\frac{2}{3}}\frac{D^i\varrho D_i\varrho}{1+\varrho}.$$

becomes

$$B^0\partial_\tau\mathbf{U}+\tau^{\gamma-\frac{7}{3}}B^i\partial_i\mathbf{U}=\frac{1}{\tau}\mathscr{B}\mathbb{P}\mathbf{U}+\frac{1}{\tau}H,$$

where $\mathbf{U}:=(u_0,u_j,u)^T$ and B^0, B^i, $\mathscr{B}$ and $\mathbb{P}$ are constant matrices, i.e.

$$B^0=\begin{pmatrix}1 & & \\ & \tilde{\kappa}\delta^{jk} & \\ & & 1\end{pmatrix},\quad B^i=\begin{pmatrix}0 & \tilde{\kappa}\delta^{ij} & 0\\ \tilde{\kappa}\delta^{ik} & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\mathscr{B}=\begin{pmatrix}\frac{5}{3} & & \\ & \tilde{\kappa}(\gamma-\frac{2}{3})\delta^{ik} & \\ & & \frac{5}{3}\end{pmatrix},\quad \mathbb{P}=\begin{pmatrix}\frac{3}{5} & 0 & -\frac{\sqrt{6}}{5}\\ 0 & \delta_i^j & 0\\ -\frac{\sqrt{6}}{5} & 0 & \frac{2}{5}\end{pmatrix},$$

$$H=\left(-\frac{2\tilde{\kappa}(\gamma-1)\delta^{ij}u_iu_j}{\sqrt{6}u+\beta+2\tau^{\frac{2}{3}}},0,0\right)^T.$$

A reduced model of nonlinear Jeans instability

Question

$$\boxed{\Box\varrho(x^\mu) + \frac{a}{t}\partial_t\varrho(x^\mu) - \frac{b}{t^2}\varrho(x^\mu)(1+\varrho(x^\mu)) - \frac{c-k}{1+\varrho(x^\mu)}(\partial_t\varrho(x^\mu))^2 = kF(t),}$$

$$\varrho|_{t=t_0} = \mathring{\varrho}(x^i) > 0 \quad \text{and} \quad \partial_t\varrho|_{t=t_0} = \mathring{\varrho}_0(x^i) > 0,$$

where $\Box := \partial_t^2 - \Delta_g = \partial_t^2 - g^{ij}(t)\partial_i\partial_j$,

$$a > 1, \quad b > 0, \quad 1 < c < 3/2$$

$$3c - \sqrt{2}\sqrt{8c-5} < k < 3c + \sqrt{2}\sqrt{8c-5}.$$

$$g^{ij}(t) := \frac{m^2(\partial_t f(t))^2}{(1+f(t))^2}\delta^{ij} \quad \text{and} \quad F(t) := \frac{(\partial_t f(t))^2}{1+f(t)},$$

where $m \in \mathbb{R}$ is a given constant and $f(t)$ solves an ODE,

$$\boxed{\partial_t^2 f(t) + \frac{a}{t}\partial_t f(t) - \frac{b}{t^2}f(t)(1+f(t)) - \frac{c}{1+f(t)}(\partial_t f(t))^2 = 0,}$$

$$f(t_0) = \mathring{f} > 0 \quad \text{and} \quad \partial_t f(t_0) = \mathring{f}_0 > 0.$$

The solutions of ODEs

$$\partial_t^2 f(t) + \frac{a}{t}\partial_t f(t) - \frac{b}{t^2} f(t)(1+f(t)) - \frac{c}{1+f(t)}(\partial_t f(t))^2 = 0,$$
$$f(t_0) = \mathring{f} > 0 \quad \text{and} \quad \partial_t f(t_0) = \mathring{f}_0 > 0.$$

Theorem

1. $t_\star \in [0, \infty)$ exists and $t_\star > t_0$;
2. there is a constant $t_m \in [t_\star, \infty]$, such that there is a unique solution $f \in C^2([t_0, t_m))$ to the ODE, and
$$\lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty.$$
3. f satisfies upper and lower bound estimates,
$$1 + f(t) > \exp\left(\mathtt{C} t^{\frac{\bar{a}+\triangle}{2}} + \mathtt{D} t^{-1}\right) \qquad \text{for} \quad t \in (t_0, t_m);$$
$$1 + f(t) < \left(\mathtt{A} t^{\frac{\bar{a}-\triangle}{2}} + \mathtt{B} t^{\frac{\bar{a}+\triangle}{2}} + 1\right)^{-1} \qquad \text{for} \quad t \in (t_0, t_\star).$$

$$
\partial_t^2 f(t) + \frac{a}{t}\partial_t f(t) - \frac{b}{t^2} f(t)(1+f(t)) - \frac{c}{1+f(t)}(\partial_t f(t))^2 = 0,
$$
$$
f(t_0) = \mathring{f} > 0 \quad \text{and} \quad \partial_t f(t_0) = \mathring{f}_0 > 0.
$$

Theorem

Furthermore, if the initial data satisfies $\mathring{f}_0 > \bar{a}(1+\mathring{f})/(\bar{c}t_0)$, then

4. $t_\star$ and $t^\star$ exist and finite, and $t_0 < t_\star < t^\star < \infty$;
5. there is a finite time $t_m \in [t_\star, t^\star)$, such that there is a solution $f \in C^2([t_0, t_m))$ to the ODE, and
$$
\lim_{t\to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t\to t_m} f_0(t) = +\infty.
$$
6. the solution f has improved lower bound estimates, for $t \in (t_0, t_m)$,
$$
(1+\mathring{f})\left(1 - \mathtt{E}t_0^{\bar{a}} + \mathtt{E}t^{\bar{a}}\right)^{1/\bar{c}} < 1 + f(t).
$$

The solutions to the PDEs

$$\boxed{\Box\varrho(x^\mu)+\frac{\mathfrak{a}}{t}\partial_t\varrho(x^\mu)-\frac{\mathfrak{b}}{t^2}\varrho(x^\mu)(1+\varrho(x^\mu))-\frac{c-\mathfrak{k}}{1+\varrho(x^\mu)}(\partial_t\varrho(x^\mu))^2=\mathfrak{k}F(t),}$$

$$\varrho|_{t=t_0}=\mathring{\varrho}(x^i)>0 \quad \text{and} \quad \partial_t\varrho|_{t=t_0}=\mathring{\varrho}_0(x^i)>0,$$

Theorem

Suppose $s\in\mathbb{Z}_{\geq\frac{n}{2}+3}$, and assume $t_m>t_0$ such that $[t_0,t_m)$ is the maximal interval of existence of f. Then there are small constants $\sigma_\star,\sigma>0$, such that if the initial data satisfies

$$\left\|\frac{\mathring{\varrho}}{\mathring{f}}-1\right\|_{H^s(\mathbb{T}^n)}+\left\|\frac{\mathring{\varrho}_0}{\mathring{f}_0}-1\right\|_{H^s(\mathbb{T}^n)}+\left\|\frac{\mathfrak{m}\mathring{\varrho}_i}{1+\mathring{f}}\right\|_{H^s(\mathbb{T}^n)}\leq\frac{1}{2}\sigma_\star\sigma,$$

then there is a solution $\varrho\in C^2([t_0,t_m)\times\mathbb{T}^n)$ to the PDEs and ϱ satisfies the estimate

$$\left\|\frac{\varrho(t)}{f(t)}-1\right\|_{H^s(\mathbb{T}^n)}+\left\|\frac{\partial_t\varrho(t)}{f_0(t)}-1\right\|_{H^s(\mathbb{T}^n)}+\left\|\frac{\mathfrak{m}\partial_i\varrho(t)}{1+f(t)}\right\|_{H^s(\mathbb{T}^n)}\leq C\sigma<1$$

for $t\in[t_0,t_m)$ and some constant $C>0$. Moreover, ϱ blowups at $t=t_m$, i.e.,

$$\lim_{t\rightarrow t_m}\varrho(t,x^i)=+\infty \quad \textit{and} \quad \lim_{t\rightarrow t_m}\varrho_0(t,x^i)=+\infty,$$

with the rate estimates $(1-C\sigma)f\leq\varrho\leq(1+C\sigma)f$ and $(1-C\sigma)f_0\leq\varrho_0\leq(1+C\sigma)f_0$ for $t\in[t_0,t_m)$.

Estimates of $f(t)$

Starting from

$$\partial_t^2 f(t) + \frac{a}{t}\partial_t f(t) - \frac{b}{t^2} f(t)(1 + f(t)) - \frac{c}{1 + f(t)}(\partial_t f(t))^2 = 0$$

We construct a series of differential inequalitites in the form of

$$\partial_t G < \frac{k}{t} G + F \quad \text{or} \quad \partial_t G > \frac{k}{t} G + F$$

to estimate variants of f and $\partial_t f$ (Not sharp! but enough!).

Estimates of $\varrho(x^\mu)$

- Methods: Fuchsian formulations.
- The compactified time

$$\tau := -g(t) = -\exp\Big(-A\int_{t_0}^{t}\frac{f(s)(f(s)+1)}{s^2 f_0(s)}ds\Big)$$
$$= -\Big(1+\mathcal{b}B\int_{t_0}^{t} s^{\mathcal{a}-2}f(s)(1+f(s))^{1-\mathcal{c}}ds\Big)^{-\frac{A}{\mathcal{b}}} \in [-1,0),$$

Fuchsian fields

$$
\begin{aligned}
w(t,x^i) &:= \varrho(t,x^i) - f(t),\\
w_0(t,x^i) &:= \partial_t w(t,x^i) = \partial_t \varrho(t,x^i) - f_0(t),\\
w_i(t,x^i) &:= \partial_i w(t,x^i) = \partial_i \varrho(t,x^i).
\end{aligned}
$$

and

$$
\begin{aligned}
u(t,x^i) =& \frac{1}{f(t)} w(t,x^i),\\
u_0(t,x^i) =& \frac{1}{f_0(t)} w_0(t,x^i),\\
u_i(t,x^i) =& \frac{m}{1+f(t)} w_i(t,x^i).
\end{aligned}
$$

then

$$
\underline{u}(\tau,x^i) = u(g^{-1}(-\tau),x^i), \quad \underline{u_0}(\tau,x^i) = u_0(g^{-1}(-\tau),x^i)
$$
$$
\underline{u_i}(\tau,x^i) = u_i(g^{-1}(-\tau),x^i).
$$

Singular and regular in τ terms

Define a quantity

$$\chi(t) := \frac{t^{2-\mathscr{a}} f_0(t)}{(1+f(t))^{2-\mathscr{c}} f(t) g^{\frac{\mathscr{b}}{A}}(t)} = \frac{g^{-\frac{2\mathscr{b}}{A}}(t) t^{2(1-\mathscr{a})}}{Bf(t)(1+f(t))^{2(1-\mathscr{c})}}.$$

Then there is a function $\mathfrak{G} \in C^1([t_0, t_m))$, such that for $t \in [t_0, t_m)$,

$$\chi(t) = \frac{2\mathscr{b}B}{3-2\mathscr{c}} + \mathfrak{G}(t).$$

where $\lim_{t\to t_m} \mathfrak{G}(t) = 0$. Moreover, there is a constant $C_\chi > 0$ such that $0 < \chi(t) \leq C_\chi$ in $[t_0, t_m)$, and there are continuous extensions of χ and $\mathfrak{G}$ such that $\chi \in C^0([t_0, t_m])$ and $\mathfrak{G} \in C^0([t_0, t_m])$ by letting $\chi(t_m) := 2\mathscr{b}B/(3-2\mathscr{c})$ and $\mathfrak{G}(t_m) := 0$.

Define a quantity

$$\xi(t) := 1/[g(t)(1+f(t))],$$

then $\xi \in C^1([t_0, t_m))$ and

$$\lim_{t \to t_m} \xi(t) = 0.$$

Moreover, there is a constant $C_\star > 0$, such that $0 < \xi(t) \leq C_\star$ for every $t \in [t_0, t_m)$, and there is a continuous extension of ξ such that $\xi \in C^0([t_0, t_m])$ by letting $\xi(t_m) := 0$.

Remark

$\chi(t)$ and $\xi(t)$ help distinguish the singular term $\frac{1}{\tau}\mathbf{BP}u$ and the regular term G in the Fuchsian system:

$$B^\mu \partial_\mu u = \frac{1}{\tau}\mathbf{BP}u + G.$$

Fuchsian formulations

$$\mathscr{B}^0\partial_\tau\mathcal{U} + \mathscr{B}^j\partial_j\mathcal{U} = \frac{1}{\tau}\mathfrak{B}\mathbb{P}\mathcal{U} + \mathscr{H}$$

where $\mathcal{U} := (\underline{u_0}, \underline{u_i}, \underline{u})^T$, $\mathscr{H} := \mathscr{H}(\tau, \underline{u_0}, \underline{u}) = \left(-\underline{\mathfrak{L}}(\tau, \underline{u}), 0, -\underline{\mathfrak{K}}(\tau, \underline{u_0}, \underline{u})\right)^T$, $\mathbb{P} := \mathbb{1}$,

$$\mathscr{B}^0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta^{ki} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathscr{B}^i := \frac{m\underline{\chi}}{AB\tau}\begin{pmatrix} 0 & \delta^{ij} & 0 \\ \delta^{kj} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{B} := \frac{1}{A}\begin{pmatrix} \mathcal{b} + (2\mathcal{k} - \mathcal{c})(\frac{2\mathcal{b}}{3-2\mathcal{c}} + \frac{\underline{\mathfrak{G}}}{B}) + \underline{\mathfrak{H}} & 0 & -2\mathcal{b} + (\mathcal{c} - \mathcal{k})(\frac{2\mathcal{b}}{3-2\mathcal{c}} + \frac{\underline{\mathfrak{G}}}{B}) + \underline{\mathfrak{F}} \\ 0 & (\frac{2\mathcal{b}}{3-2\mathcal{c}} + \frac{\underline{\mathfrak{G}}}{B})\delta^{ki} & 0 \\ -(\frac{2\mathcal{b}}{3-2\mathcal{c}} + \frac{\underline{\mathfrak{G}}}{B}) & 0 & \frac{2\mathcal{b}}{3-2\mathcal{c}} + \frac{\underline{\mathfrak{G}}}{B} \end{pmatrix},$$

Nonlinear gravitational instabilities for expanding Newtonian universes with inhomogeneous pressure and entropy

The dimensionless and normalized Euler–Poisson system

$$\partial_t \rho + \partial_i(\rho v^i) = 0,$$
$$\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0,$$
$$\partial_t s + v^i \partial_i s = 0,$$
$$\Delta\phi = \delta^{ij}\partial_i\partial_j\phi = 4\pi\rho.$$

The *equation of state* becomes

$$p = K e^s \rho^{\frac{4}{3}} + \mathfrak{p}, \quad \text{for } K \geq 0.$$

The initial data at $t = 1$ is given by

$$\rho|_{t=1} = \frac{\iota^3}{6\pi}, \quad v^i|_{t=1} = \frac{2}{3}x^i \quad \text{and} \quad s|_{t=1} = \ln(\delta_{kl}x^k x^l)^{\mathrm{sgn}(1-\iota^3)}.$$

where ι satisfies an important identity

$$\iota^3 + 9\Big(\frac{\tilde{K}}{6}\Big)^{\frac{1}{3}}\iota - 1 = 0 \Rightarrow \iota \in (0, 1]$$

- Note $\tilde{K}$ and ι are *dimensionless constants* depending on the *molar mass* of the fluids and the distributions of the entropy or temperature.

- If $\iota \neq 1$, the data of the entropy implies the initial distribution of the temperature $\mathcal{T} \propto |\boldsymbol{x}|^2$.

Background solutions

There is an exact solution on $(t, x^k) \in [t_0, \infty) \times \mathbb{R}^3$,

$$\mathring{\rho}(t) = \frac{\iota^3}{6\pi t^2}, \quad \mathring{p}(t) = K t^{-\frac{4}{3}} \delta_{kl} x^k x^l \mathring{\rho}^{\frac{4}{3}} + \mathfrak{p}, \quad \mathring{v}^i(t, x^k) = \frac{2}{3t} x^i,$$
$$\mathring{\phi}(t, x^k) = \frac{2}{3}\pi \mathring{\rho} \delta_{ij} x^i x^j = \frac{\iota^3}{9t^2} \delta_{ij} x^i x^j, \quad \mathring{s}(t, x^k) = \ln(t^{-\frac{4}{3}} \delta_{kl} x^k x^l)^{\mathrm{sgn}(1-\iota^3)},$$

Homogeneous perturbations in following sense

A homogeneous initial perturbations and characterized by two positive parameters β and γ in the following ways,

$$\rho|_{t=1} = (1+\beta)\frac{\iota^3}{6\pi}, \quad v^i|_{t=1} = \Big(\frac{2}{3} - \gamma\Big)x^i,$$
$$s|_{t=1} = \ln\Big((1+\beta)^{\frac{2}{3}}\delta_{kl}x^k x^l\Big)^{\mathrm{sgn}(1-\iota^3)}.$$

We construct solutions

$$\rho(t) = (1+f(t))\mathring{\rho}(t) = \frac{\iota^3(1+f(t))}{6\pi t^2},$$

$$v^i(t,x^i) = \frac{2}{3t}x^i - \frac{f'(t)}{3(1+f(t))}x^i,$$

$$\phi(t,x^i) = \frac{2}{3}\pi\mathring{\rho}(1+f(t))|\boldsymbol{x}|^2 = \frac{\iota^3(1+f(t))|\boldsymbol{x}|^2}{9t^2},$$

$$s(t,x^k) = \ln\Big(t^{-\frac{4}{3}}(1+f)^{\frac{2}{3}}\delta_{kl}x^kx^l\Big)^{\mathrm{sgn}(1-\iota^3)}.$$

and the *density contrast* $\varrho(t) = f(t)$ where $|\boldsymbol{x}|^2 := \delta_{ij}x^ix^j$ and $f(t)$ is a solution of the following nonlinear ODE,

$$f''(t) + \frac{4}{3t}f'(t) - \frac{2}{3t^2}f(t)(1+f(t)) - \frac{4(f'(t))^2}{3(1+f(t))} = 0,$$

$$f|_{t=t_0} = \beta \quad \text{and} \quad f'|_{t=t_0} = 3(1+\beta)\gamma.$$

Moreover, the pressure becomes $p(t) = \frac{K\iota^4}{(6\pi)^{\frac{4}{3}}t^4}(1+f)^2\delta_{kl}x^kx^l$.

Nonlinear gravitational instabilities for expanding spherical symmetric Newtonian universes with inhomogeneous density and pressure

The dimensionless and normalized Euler–Poisson system

$$\begin{aligned}\partial_t \rho + \partial_i(\rho v^i) =& 0,\\ \partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi =& \mathscr{D}^i(t, x^j, \rho, v^k, s, \phi),\\ \partial_t s + v^i \partial_i s =& \mathscr{S}(t, x^j, \rho, v^k, s, \phi),\\ \Delta\phi = \delta^{ij}\partial_i\partial_j \phi =& 4\pi\rho.\end{aligned}$$

EoS is

$$p = K e^s \rho^{\frac{4}{3}} \quad \text{for} \quad K > 0.$$

The *initial data* is given by *spherical symmetric functions* $\beta\mathscr{d}(|\mathbf{x}|)$ and $\gamma\mathfrak{v}(|\mathbf{x}|)$,

$$\begin{aligned}\rho|_{t=1}(x^i) = \rho_0(|\mathbf{x}|) :=& \frac{\iota^3}{6\pi}(1 + \beta\mathscr{d}(|\mathbf{x}|)),\\ v^i|_{t=1}(x^i) = v_0^i(|\mathbf{x}|) :=& \frac{2}{3}x^i + \gamma\mathfrak{v}(|\mathbf{x}|)x^i,\\ s|_{t=1}(x^i) = s_0(|\mathbf{x}|) :=& \ln\Big(\frac{(1+\beta\mathscr{d}(|\mathbf{x}|))^{\frac{2}{3}+\omega}}{(1+\beta)^\omega}|\mathbf{x}|^2\Big).\end{aligned}$$

Assumptions

1. Dampings and entropy productions:

$$\mathscr{D}^i(t, x^j, \rho, v^k, s, \phi) := -\frac{\kappa f_0}{1+f}\left[v^i - \left(\frac{2}{3t} - \frac{f_0}{3(1+f)}\right)x^i\right],$$

$$\mathscr{S}(t, x^j, \rho, v^k, s, \phi) := -\left(\frac{2}{3}+\omega\right)\partial_i v^i + \frac{2v^i x_i}{|\mathbf{x}|^2} + 3\omega\left(\frac{2}{3t} - \frac{f_0}{3(1+f)}\right),$$

$\mathscr{D}^i$ serves as a damping term that arises directly from the inhomogeneous densities, while $\mathscr{S}$ accentuates the growth of entropy caused by the aforementioned inhomogeneities through a flow of matter that depends on temperature.

2. Periodicity and spherical symmetry of data: To simplify the analysis, we assume that the initial data $\mathscr{\ell}(|\mathbf{x}|)$ and $\mathscr{v}(|\mathbf{x}|)$ are both 1-*log-periodic* functions, as defined below.

Definition

A function $F(|\mathbf{x}|)$ is called t-**log-periodic** if there is a t-parameterized exp-log transform y_t satisfying

$$|\mathbf{x}| = y_t(\zeta) := t^{\frac{2}{3}}\big(1+f(t)\big)^{-\frac{1}{3}} \exp\zeta \;\Leftrightarrow\; \zeta = y_t^{-1}(|\mathbf{x}|) = \ln\Big(t^{-\frac{2}{3}}\big(1+f(t)\big)^{\frac{1}{3}}|\mathbf{x}|\Big),$$

such that $\acute{F}(\zeta) := F \circ y_t(\zeta)$ is a periodic function with the unit period, that is, $\acute{F}(\zeta + m) = \acute{F}(\zeta)$ for any $m \in \mathbb{Z}$ and $\zeta \in \mathbb{R}$.

Physical interpretations of $\mathscr{D}^i$ and $\mathscr{S}$

$$\mathscr{D}^i(t, x^j, \rho, v^k, s, \phi) = \underbrace{-\frac{\kappa f_0}{1+f}\check{v}^i}_{\text{damping due to inhomogeneous densities}},$$

$$\mathscr{S}(t, x^j, \rho, v^k, s, \phi) = \underbrace{-\frac{1}{\rho}\partial_i\Big(\frac{\mathfrak{J}^i}{\mathfrak{T}}\Big)}_{\text{flux of entropy}} + \underbrace{\frac{\mathfrak{J}^i\mathfrak{F}_i}{\mathfrak{T}\rho}}_{\text{entropy productions}}.$$

The temperature distribution is given by

$$\mathfrak{T} \propto |\mathbf{x}|^2, \quad \text{i.e.,} \quad \mathfrak{T} = \Bbbk(t)\delta_{kl}x^k x^l, \quad \text{for some time function } \Bbbk(t).$$

The *entropy flux* and *thermodynamic force* are given, respectively, by

$$\mathfrak{J}^i := \Big(\frac{2}{3}+\omega\Big)\mathfrak{T}\rho\check{v}^i \quad \text{and} \quad \mathfrak{F}_i := \partial_i \ln\big(\rho\mathfrak{T}^{\frac{1}{\frac{2}{3}+\omega}}\big).$$

Recall two solutions

Newtonian universes (backgroud solutions): If $\beta = \gamma = 0$, then the initial data reduce to

$$\rho|_{t=1}(x^i) = \frac{\iota^3}{6\pi}, \quad v^i|_{t=1}(x^i) = \frac{2}{3}x^i \quad \text{and} \quad s|_{t=1}(x^i) = \ln|\mathbf{x}|^2.$$

Then the solution to the Euler–Possion system is

$$\mathring{\rho}(t) = \frac{\iota^3}{6\pi t^2}, \quad \mathring{p}(t) = Kt^{-\frac{4}{3}}\delta_{kl}x^k x^l \mathring{\rho}^{\frac{4}{3}}, \quad \mathring{v}^i(t, x^k) = \frac{2}{3t}x^i,$$
$$\mathring{\phi}(t, x^k) = \frac{2}{3}\pi\mathring{\rho}\delta_{ij}x^i x^j = \frac{\iota^3}{9t^2}\delta_{ij}x^i x^j \quad \text{and} \quad \mathring{s}(t, x^k) = \ln(t^{-\frac{4}{3}}\delta_{kl}x^k x^l).$$

Homogeneous blowup solutions (reference solutions): If constants $\beta > 0$, $\gamma > 0$ and $\mathscr{d}(|\mathbf{x}|) = 1$, $\mathscr{v}(|\mathbf{x}|) = -1$, then the initial data becomes

$$\rho|_{t=1}(x^i) = \frac{\iota^3}{6\pi}(1+\beta), \quad v^i|_{t=1}(x^i) = \Big(\frac{2}{3} - \gamma\Big)x^i,$$
$$s|_{t=1}(x^i) = \ln\big((1+\beta)^{\frac{2}{3}}|\mathbf{x}|^2\big).$$

There is a solution to the Euler–Poisson system given by

$$\rho_r(t) = \frac{\iota^3(1+f(t))}{6\pi t^2}, \quad v_r^i(t,x^i) = \frac{2}{3t}x^i - \frac{f'(t)}{3(1+f(t))}x^i,$$
$$\phi_r(t,x^i) = \frac{\iota^3(1+f(t))|\boldsymbol{x}|^2}{9t^2} \quad \text{and} \quad s_r(t,x^k) = \ln\big(t^{-\frac{4}{3}}(1+f)^{\frac{2}{3}}\delta_{kl}x^k x^l\big),$$

and the density contrast $\varrho_r(t) = f(t)$.

Instabilities

A homogeneous initial perturbation around $(\mathring{\rho}, \mathring{v}^i, \mathring{\phi}, \mathring{s})$ results in a blowup solution $(\rho_r, v_r^i, \phi_r, s_r)$, meaning that the Newtonian universe $(\mathring{\rho}, \mathring{v}^i, \mathring{\phi}, \mathring{s})$ is gravitationally unstable.

Main theorem

Theorem

Under assumptions, suppose $s \in \mathbb{Z}_{>\frac{7}{2}}$, $\iota^3 \in (0, 1/5]$ and $f \in C^2([1, t_m))$ solves the ODE where $\beta > 0$ and $\gamma > 0$, and assume $t_m > 1$ such that $[1, t_m)$ is the maximal interval of existence of f. Then there are small constants $\sigma_\star, \sigma > 0$, such that if the initial data satisfies

$$\left\| \mathscr{d} \circ y_1 - 1 \right\|_{H^{s+1}(\mathbb{T})} + \left\| \mathscr{v} \circ y_1 + 1 \right\|_{H^{s+1}(\mathbb{T})} \leq \sigma_\star \sigma,$$

Theorem

then

1. *there is a solution* $(\rho, v^i, s, \phi) \in C^2([1, t_m) \times \mathbb{R}^3)$ *to the system and* $\rho(t, |\mathbf{x}|)$, $v^i(t, |\mathbf{x}|)x_i/|\mathbf{x}|^2$ *are t-log-periodic and spherical symmetric;*
2. *there is a constant* $C_1 \in (0, 1/\sigma)$, *such that* ϱ *and* v^i *satisfy the estimates*

$$0 < (1 - C_1\sigma)f(t) < \varrho(t, x^i) < (1 + C_1\sigma)f(t),$$
$$0 < (1 - C_1\sigma)f_0(t) \le \partial_t\varrho(t, x^i) \le (1 + C_1\sigma)f_0(t),$$
$$-C\sigma(1 + f(t)) \le x^i\partial_i\varrho(t, x^i) \le C\sigma(1 + f(t))$$

and

$$\Big(\frac{2}{3t} - \frac{(1 + C_1\sigma)f_0(t)}{3(1 + f(t))}\Big)x^i < v^i(t, x^i) < \Big(\frac{2}{3t} - \frac{(1 - C_1\sigma)f_0(t)}{3(1 + f(t))}\Big)x^i$$

for $(t, x^i) \in [1, t_m) \times \mathbb{R}^3$;

3. *the entropy* s *can be expressed by*

$$s = \ln\Big(t^{-\frac{4}{3}}\frac{(1 + \varrho)^{-\frac{14}{15}}}{(1 + f)^{-\frac{8}{5}}}\delta_{kl}x^k x^l\Big);$$

4. ϱ *and* $\partial_t\varrho$ *both blowup at* $t = t_m$, *i.e.,*

$$\lim_{t\to t_m}\varrho(t, x^i) = +\infty \quad \text{and} \quad \lim_{t\to t_m}\partial_t\varrho(t, x^i) = +\infty;$$

5. *if the parameter* γ *of the data satisfies* $\gamma > 1/3$, *then there is a finite time* $t_m < \infty$, *such that the density contrast* ϱ *and its derivative* $\partial_t\varrho$ *blow up at a finite time* t_m.

Methods

1. Fuchsian formulations.
2. The compactified time transformation:

$$g(t) := \exp\Big(-A\int_1^t \frac{f(s)(f(s)+1)}{s^2 f_0(s)} ds\Big) > 0$$
$$\tau := -g(t) \in [-1, 0),$$

Basic ideas

- Similar to Jeans

$$\left.\begin{aligned} \mathrm{tr}\partial_{X^k}[\text{momentum conservations}]_{(t,X^i)} &\Rightarrow \text{the eq. of } \partial_t\Theta, \\ \text{the continuity eq. } &\Rightarrow \text{the expression of } \Theta \end{aligned}\right\} \Rightarrow \begin{array}{l}\text{The 2nd order}\\ \text{hyperbolic eq.}\end{array}$$

- Firstly, we use comoving coordinates with the reference solutions to rewrite the system.
- Velocity transform yeilds a rescaled speed ν

$$\underset{\sim}{\check{v}}^i(t, X^k) = \frac{f_0}{3(1+f)}\nu(t,R)X^i$$

- Log-periodic coordinate

$$\mathbb{R} \ni \zeta := \ln R \quad \text{i.e., } R = e^\zeta \quad \text{where } R \in (0,\infty).$$

- Continuity equation plays a very important role bridging density and speed ν to construct the Fuchsian formulations.

$$\Theta = \frac{f_0}{1+f} - \frac{\partial_t \hat{\varrho}}{1+\hat{\varrho}} - \frac{f_0}{3(1+f)}\frac{\nu R\partial_R\hat{\varrho}}{1+\hat{\varrho}} = \frac{f_0}{1+f}\nu + \frac{f_0}{3(1+f)}R\partial_R\nu.$$

Eventually, we arrive at

$$\Box_g \hat{\varrho} + \Big(\frac{4}{3t} + \frac{\kappa f_0}{1+f}\Big)\partial_t \hat{\varrho} - \frac{2}{3t^2}\hat{\varrho}(1+\hat{\varrho}) - \frac{4(\partial_t \hat{\varrho})^2}{3(1+\hat{\varrho})} = F_1,$$
$$\partial_t \nu + \frac{f_0}{3(1+f)}\nu \partial_\zeta \nu = G_1,$$

where the wave operator is

$$\Box_g := \partial_t^2 - g^{\zeta\zeta}\partial_\zeta^2 + 2g^{0\zeta}\partial_\zeta\partial_t,$$
$$g^{\zeta\zeta} := \frac{(2+\omega)(1-\iota^3)}{9t^2}\frac{(1+\hat{\varrho})^{\omega+1}}{(1+f)^\omega} - \frac{f_0^2}{9(1+f)^2}\nu^2, \quad g^{0\zeta} := \frac{f_0}{3(1+f)}\nu,$$

Fuchsian fields

$$
\begin{aligned}
w(t,\zeta) &:= \hat{\varrho}(t,\zeta) - f(t), \\
w_0(t,\zeta) &:= \partial_t w(t,\zeta) = \partial_t \hat{\varrho}(t,\zeta) - f_0(t), \\
w_\zeta(t,\zeta) &:= \partial_\zeta w(t,\zeta) = \partial_\zeta \hat{\varrho}(t,\zeta).
\end{aligned}
$$

$$
u(t,\zeta) = \frac{1}{f(t)} w(t,\zeta), \quad u_0(t,\zeta) = \frac{1}{f_0(t)} w_0(t,\zeta)
$$
$$
u_\zeta(t,\zeta) = \frac{c}{1+f(t)} w_\zeta(t,\zeta).
$$

$$
\underline{u}(\tau,\zeta) = u(g^{-1}(-\tau),\zeta), \quad \underline{u_0}(\tau,\zeta) = u_0(g^{-1}(-\tau),\xi)
$$
$$
\underline{u_\zeta}(\tau,\zeta) = u_\zeta(g^{-1}(-\tau),\zeta).
$$

Evolutions of "gravity"

Introduce a new gravity

$$\Psi(t,\zeta) = \frac{1}{e^{3\zeta}} \int_{-\infty}^{\zeta} u(t,z) e^{3z} dz.$$

By continuity equation, we can put into an evolution equation

$$\partial_t \Psi = -\frac{f_0}{f(1+f)}\Psi - \frac{f_0}{3f}\Big(1 + \frac{fu}{1+f}\Big)\nu.$$

Fuchsian formulations

$$\mathscr{B}^0\partial_\tau \mathcal{U} + \mathscr{B}^\zeta \partial_\zeta \mathcal{U} = \frac{1}{\tau}\mathfrak{B}\mathbb{P}\mathcal{U} + \mathscr{H} + (-\tau)^{-\frac{1}{2}}\mathscr{F},$$

where $\mathbb{P} = \mathbb{I}$,

$$\mathscr{B}^0 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4+B^{-1}\underline{\mathfrak{G}}}\left(\frac{25}{36} + \frac{25}{36\underline{f}} + \mathfrak{z}_0\right) & 0 & 0 & 0 \\ 0 & 0 & \lambda + \frac{1}{30}(3-8\iota^3) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathscr{B}^\zeta := \frac{1}{A\tau}\begin{pmatrix} -\frac{2\underline{\chi}}{3B}\underline{\nu} & \frac{5}{36} + \frac{5}{36\underline{f}} + \frac{1}{5}\mathfrak{z}_0 & 0 & 0 & 0 \\ \frac{5}{36} + \frac{5}{36\underline{f}} + \frac{1}{5}\mathfrak{z}_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(3\iota^3+2)^2}{6(10\lambda+\iota^3+9)} \end{pmatrix},$$

$$\mathfrak{B} := \frac{1}{A}\begin{pmatrix} 4\lambda + \mathfrak{z}_1 & \mathfrak{z}_2 & \frac{2(3-8\iota^3)}{15} - 4\lambda + \mathfrak{z}_3 & \mathfrak{z}_4 & 0 \\ 0 & \frac{25}{36} + \mathfrak{z}_0 & 0 & 0 & 0 \\ -\frac{2(3-8\iota^3)}{15} - 4\lambda & 0 & 4\lambda + \frac{2(3-8\iota^3)}{15} & 0 & 0 \\ 0 & \frac{2(1-\iota^3)}{3} & -\frac{2(1-\iota^3)}{5} + \mathfrak{z}_5 & 4\lambda + 4 + \mathfrak{z}_6 & 2\iota^3 \\ 0 & 0 & -\frac{(3\iota^3+2)^2}{6(10\lambda+\iota^3+9)} & \frac{4}{3} + \mathfrak{z}_7 & \frac{(3\iota^3+2)^2}{2(10\lambda+\iota^3+9)} \end{pmatrix},$$

$$\mathscr{F} := \begin{pmatrix} -\frac{1}{AB}\left(\lambda - \frac{1}{6}\right)(-\tau)^{-\frac{1}{2}}\underline{\mathfrak{G}}(\underline{u_0} - \underline{u}) \\ 0 \\ -\frac{1}{4AB}\left(-\frac{2(3-8\iota^3)}{15} - 4\lambda\right)(-\tau)^{-\frac{1}{2}}\underline{\mathfrak{G}}(\underline{u_0} - \underline{u}) \\ -\frac{1}{AB}\left(\lambda + \frac{5}{6}\right)(-\tau)^{-\frac{1}{2}}\underline{\mathfrak{G}\nu} \\ -\frac{1}{2AB}(-\tau)^{-\frac{1}{2}}\mathfrak{G}\nu \end{pmatrix}$$

and $\mathcal{H} = \mathcal{H}(\tau, \mathcal{U}) := \big(H_1, H_2, H_3, H_4, H_5\big)^T$ such that

$$H_1 = H_1(\tau, \mathcal{U}) = -\frac{1}{A}\underline{\xi}\Big[4\lambda + \Big(\lambda - \frac{1}{6}\Big)B^{-1}\underline{\mathfrak{G}}\Big]\underline{u},$$

$$H_2 = H_2(\tau, \mathcal{U}) = -\frac{25}{36A}\underline{\xi}\Big(1 + \frac{1}{\underline{f}}\Big)\underline{u_\zeta},$$

$$H_3 = H_3(\tau, \mathcal{U}) = \Big(\lambda + \frac{3 - 8\iota^3}{30}\Big)\frac{1}{A}\underline{\xi}\Big(1 + \frac{1}{\underline{f}}\Big)(4 + B^{-1}\underline{\mathfrak{G}})(\underline{u_0} - \underline{u}),$$

$$H_4 = H_4(\tau, \mathcal{U}) = -\frac{2(1 - \iota^3)}{3A}\underline{\xi}\Big(1 + \frac{1}{\underline{f}}\Big)\Big(1 + \frac{\underline{f\,u}}{1 + \underline{f}}\Big)^{-\frac{8}{5}}\underline{u_\zeta},$$

$$H_5 = H_5(\tau, \mathcal{U}) = -\frac{\underline{\chi\xi}}{3AB}\Big(1 + \frac{1}{\underline{f}}\Big)\Big(1 + \frac{\underline{f\,u}}{1 + \underline{f}}\Big)\underline{\nu} - \frac{\underline{\chi\xi}}{AB}\Big(1 + \frac{1}{\underline{f}}\Big)\underline{\Psi}.$$

remark

A lots of improved estimates are required to verify it is a Fuchsian formulation, for example,

$$|\underline{\mathfrak{G}}(\tau)| \lesssim (-\tau)^{\frac{1}{2}} \quad \text{and} \quad \lim_{t \to t_m}\big(1/[g^2(t)(1 + f(t))]\big) = 0, \text{ etc.}$$

Thank you
for your attention!