# Sobolev spaces, Chap 1. Introduction

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**Reference**: ① L. Evans, PDE
② E. Taylor, PDE I & III
③ R. Adam, Sobolev spaces
④ H. Brezis, Functional analysis, Sobolev spaces and PDEs

**Sobolev spaces**: Functional analysis (上游) ① $\xrightarrow{\text{applied}}$ Sobolev spaces $\xrightarrow{\text{applied}}$ PDEs (下游)

tool of applying ideas of functional analysis to ② glean information concerning PDE

② Glean information:

how to research PDEs: can not directly solve the equations (Usually it's extremely difficult)

ideas to research PDEs: directly and judiciously using the expressions of the equations to arrive at certain equivalent formulations, and then using these equivalent formulations to draw certain information of the possible solutions (Usually certain statistics or estimates of the solutions)

idea may be simple, but in order to make this idea can be effectively applied, we have to delicately design suitable statistics and the way of estimates ← one delicate and effective design is the tool of

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# Sobolev spaces

① **Functional analysis**: what is the functional analysis?

impose certain structures on the set of linear functions on $V$ to make these linear functions are well-organized to form certain spaces of functions

this point of view enable one to use functional analysis and e.g. fixed pt thm, ...

$V$

vector spaces with topology (limit-related structure)

dual

$V^*$

continuous linear functionals on $V$. gathered these functions with certain structures

(dual alg. structure with dual top. structure) {weak top, strong top}

space of functions

$X$

reflexible?

leads to other better features

$V^{**}$

Conti. linear functional on $V^*$

def. + & scale multip

{strong top, weak* top}

dual

functional analysis

$A: X \to Y$

operators $A$

operator $A$ encodes the structure of the PDEs (including bdry conditions)

space of functions

$Y$

3 Fund. thm.

① Hahn-Banach thm. $V$ extension $U$ $\xrightarrow{\text{leads to}}$ elements of $V^*$ extends to elements of $U^*$

② open mapping thm: surjective $\to$ open mapping

③ Uniform bd thm: $A$ bded mapping pointwisely $\to$ $\|A\|_{op} < C$ uniformly.

In addition, compact operators $\rightsquigarrow$ Fredholm thm ...

* When studying PDE with the help of functional analysis, the difficult part is to find the "right" spaces $X$ & $Y$ and the "right" abstract operator $A$

Next, let's see how Sobolev spaces capture the statistics and certain characteristics of functions

Question: which properties Sobolev spaces can capture?

$\boxed{\text{A function space norm — rigorously quantify various statistics of a function } f: X \to \mathbb{C}}$

e.g.

$L^p(X,\mu)$ — quantify "height" & "width"

step function $f = A 1_E$

$$\|f\|_{L^p(X,\mu)} = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} = \left(\int A^p 1_E^p \, d\mu\right)^{\frac{1}{p}}$$

$$= A \left(\int 1_E^p \, d\mu\right)^{\frac{1}{p}} = \underbrace{A}_{\text{height}} \underbrace{|\mu(E)|^{\frac{1}{p}}}_{\text{width}}$$

More interesting features than "height" & "width"

singularity of a function

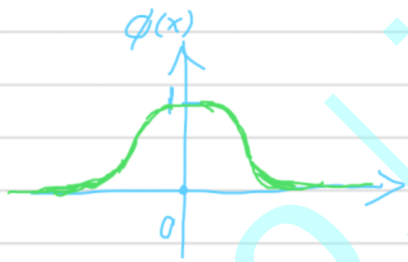
frequency scale

① regularity: how smooth a function is (how many times one can differentiate) ← Taylor expansion.

② frequency scale: how quickly the function oscillates. (inversely proportional to the wave length) ← Fourier expansion

Examples:

EX1: $\phi \in C_c^\infty(\mathbb{R})$   $\phi(x) = 1$ around $x = 0$.




$f(x) := \phi(x)\sin(Nx)$ oscillates at a wavelength of about $1/N$

$\lambda = \frac{1}{N}\cdot 2\pi \Rightarrow freq = \frac{2\pi}{\lambda} = N$

less smooth as $N \to \infty$ (more spikes and oscillations)

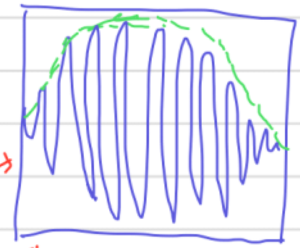
$f'(x) = \phi'(x)\sin(Nx) + N\phi(x)\cos(Nx) \nearrow \infty$ as $N \to \infty$

Key: estimate the statistics of the properties of a function (key idea of PDEs)

higher derivatives grow at even faster rates.

① no any regularity as $N \nearrow \infty$ →

② height & width are bded uniformly →



③ regularity indep. of height & width!

EX 2: $g(x) := N^{-s}\phi(x)\sin(Nx)$, $s \geq 0$

freq $= N$ hight adjust the freq. slightly to make sure it has a certain amount of regularity as $N \nearrow \infty$. derivatives of $g(x)$ show these regularities.

i.e. the kth derivatives of $g$ stays bded in $N$ as long as $k \leq s$. "s degree of regularity" as $N \nearrow \infty$.

EX3: Similarly $g(x) := N^{-s}\phi(Nx)$ ($\because \phi'(x)$ bded)

freq $= N$. ↓ $s$ degrees of regularity as $N \nearrow \infty$.

$$g'(x) = N^{-s}\phi' \cdot N = N^{1-s}\phi'$$

admits $s$ Fourier expansion to this freq.

EX4: function $\phi(x)\,|x|^s\,1_{x>0}$ s degree of regularity (diff. up to s deriv.)

dyadic decomposition of $x$.

freq $= 2^n$ s degree of regu

$$\phi(x)\,|x|^s\,1_{x\geq 0} = \sum_{n=0}^{\infty} \psi(2^n x)\,|x|^s = \sum_{n=0}^{\infty} |x|^s\left(\phi(2^n x) - \phi(2^{n+1}x)\right) 1_{x>0}$$

where $\psi(x) := (\phi(x) - \phi(2x))\,1_{x>0}$ is a bump function supported away from 0.

has a range of freq. scales ranging from 1 to $+\infty$


There are variety of function space norms that can capture frequency scale (or regularity)

e.g. Sobolev spaces $\|f\|_{W^{s,p}(\mathbb{R}^n)}$

Roughly, $W^{s,p}$ norm is like $L^p$ norm but with "s additional degree of regularity"

extra information.

For example $f(x) := A\,\phi\left(\frac{x}{R}\right)\sin(Nx)$, ($\phi$ fixed test function)

$R, N$ are large.

info: $\begin{cases} \text{height} & |A| \\ \text{width} & R \\ \text{frequ.} & N \end{cases}$

$$\|f\|_{W^{s,p}} \sim |A|\,R^{1/p}\,\underset{\uparrow}{N^s}$$

extra info.

$$\| \quad \|f\|_{L^p} + \cdots + \|\partial^s f\|_{L^p}$$

$$\cap\!| \quad |A|\,R^{\frac{1}{p}} + N\,|A|R^{\frac{1}{p}} + \cdots + |A|\,R^{\frac{1}{p}}N^s$$

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Sobolev spaces

$W^{s,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) +$ additional benefit

describing regularity (smoothness) or frequency

$\hat{f}(\xi) = \delta(\xi)$

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Uncertainty Principle $\|f(x)\|_{L^2}^2 \lesssim \|xf(x)\|_{L^2}\,\|\xi\hat{f}(\xi)\|_{L^2}$

if $\|f(x)\|_{L^2} > 0 \Rightarrow \|xf(x)\|_{L^2}\,\|\xi\hat{f}(\xi)\|_{L^2} \geq \delta > 0$

Constraint between width & frequency scale of a function

give a hint of constraint on, for instance, $N^s$ and $R^{\frac{1}{p}}$.

eventually. constraints on $s$ & $p$ $\to$ Sobolev embedding Thm

$\uparrow$

trade regularity (smoothness) for integrability

roughly, $W^{s,p} \subset W^{s',p'}$ where $s' < s$ & $p' > p$

for example. Fundamental Thm of Calculus.

$C^1 \ni f: \mathbb{R} \to \mathbb{R}$ has $f' \in L^1(\mathbb{R}) \Rightarrow \int_{\mathbb{R}} |f'| dx < \infty$

Then $\left| \int_a^x f'(s) ds \right| = |f(x) - f(a)|$

$\wedge|$

$\int_{-\infty}^{+\infty} |f'(s)| ds$ $\Rightarrow |f(x)| < |f(a)| + \|f'\|_{L^1}$

$\wedge$

$+\infty$

$\Rightarrow |f(x)| < \infty$

$\Rightarrow f \in W^{1,1} \Rightarrow f \in L^\infty$

$\Rightarrow W^{1,1} \subset L^\infty \xrightarrow{\text{iteration + Fubini}} W^{d,1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$

Sobolev norms measure a combination of 3 aspects of a function: height (amplitude), width (measure of the support) and frequency (inverse wavelength). If a function has height $A$, support is a set of volume $V$, frequency $N$, then $W^{k,p}$ norm is $\sim$ $\underline{A N^k V^{\frac{1}{p}}}$

Uncertainty principle implies that a function with frequency $N$, then the position. a wavelength $\frac{1}{N} \Rightarrow$ support $\frac{1}{N^d} \Rightarrow \boxed{V \gtrsim \frac{1}{N^d}}$ (think e.g. $f(x) = \begin{cases} 1 & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases}$)

position, a wavelength $\frac{1}{N}$ $\Rightarrow$ support $\frac{1}{N^d}$ $\Rightarrow$ $\boxed{V \gtrsim \frac{1}{N^d}}$ (think e.g. $f(x) = \begin{cases} 1 & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases}$)

$N \gtrsim \frac{1}{V^{\frac{1}{d}}} = V^{-\frac{1}{d}}$

encode most of the content of the Sobolev embedding thm, except for endpoint.

$\boxed{\text{Trade regularity for integrability}}$

$A N^k V^{\frac{1}{p}} \gtrsim A N^{k-\frac{d}{p}}$

$\Downarrow$

$W^{k,p} \subset W^{k-\frac{d}{p},\infty}$

$$A N^k V^{\frac{1}{p}} = A V^{\frac{1}{p}-\frac{k}{d}}$$

$W^{k,p} \subset L^{\frac{1}{\frac{1}{p}-\frac{k}{d}}} = L^{p^*}$ where $p^* = \frac{1}{\frac{1}{p}-\frac{k}{d}}$

$\Leftrightarrow$ $\boxed{\frac{1}{p^*} = \frac{1}{p} - \frac{k}{d}}$

Hölder spaces & Sobolev spaces

Monday, December 30, 2019 8:47 AM

$\boxed{\text{Hölder spaces}}$: ① well-known Lipschitz continuity

Assume $U \subset \mathbb{R}^n$ is open and $0 < \gamma < 1$, we have a useful class of Lipschitz continuous functions $u : U \to \mathbb{R}$ given by

$$|u(x) - u(y)| < C|x-y|$$

for any $x, y \in U$ and some constant $C > 0$.

② generalize to Hölder continuity

change above $|u(x) - u(y)| < |x-y|$ to be $\boxed{|u(x) - u(y)| < C|x-y|^{\gamma} \text{ for some } \gamma \in (0,1]}$

called Hölder continuous with exponent $\gamma$

Remark: change $|x-y|$ or $|x-y|^{\gamma}$ to even more general function $\omega : [0,\infty] \to [0,\infty]$, then inequality can be replaced by

$$|u(x) - u(y)| < \omega(|x-y|)$$

$\omega$ is called modulus of continuity

a general tool to measure quantitatively the uniform continuity.

Def: (1) $\|u\|_{C(\bar{U})} := \sup_{x \in U} |u(x)|$

provided $u : U \to \mathbb{R}$ is bounded & conti.

(2) $\gamma$th-Hölder semi-norm of $u : U \to \mathbb{R}$ is

Since it's 0 for constant functions.

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{u(x)-u(y)}{|x-y|^\gamma} \right\}$$

and the $\gamma$th-Hölder norm is

rule out the constants

$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$$

def by iteration

**Def:** The Hölder space $C^{k,\gamma}(\bar{U})$ consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}$$

is finite.

Useful space due to its completeness (every limiting pt are in this space)

no leaking pt.

**Thm:** $C^{k,\gamma}(\bar{U})$ is a Banach Space.

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Sobolev spaces (good)

Hölder spaces (bad) are not proper spaces capturing solutions of PDEs usually. Since we can not get decent estimates to conclude that solutions of PDEs are in such spaces.

drawbacks of Hölder spaces: too smooth.

PDE $\to$ $\boxed{A: X \to Y}$

$X, Y$ takes $C^{k,\gamma}$ $\to$ no good estimates to ensure this

$\boxed{\text{Find suitable function space \& suitable topology on it}}$

Key drawback is the topology is too strong, which leads the limiting process is more difficult to

& suitable topology on it | leads the limiting process is more difficult to achieve $\Leftarrow$ Hard to find solutions

idea to overcome this difficulty: Find (generalized) solutions in weaker topology. Since limiting process is much easier to achieve, one may find "weak" solutions which are not the classical solutions.

If one can prove the "weak" solution has better regularities, they may become the classical solution.

Therefore, the first key step is to find the candidates of solutions in weaker topology ("weak" solutions)

Sobolev spaces provide a suitable function space with suitable weak topology

$\boxed{\text{Weak derivatives}}$

Use an example to see how to use weak topology to search weak solutions (candidates of solutions)

weak formulation of eq. 比原方程更弱的选拔标准选取候选人.

Concepts:

~ weak derivatives and weak solutions.

~ weak derivatives and weak solutions.

EX:

Given $f \in C([a,b])$, find a function $u$ satisfying

$$(*) \begin{cases} -u''+u=f & \text{on } [a,b] \quad (*1) \\ u(a)=u(b)=0 & (*2) \end{cases}$$

← can be solved explicitly
ingore this and see how to estimate the features of solution to get it.

A classical (strong) solution is a $C^2$ function on $[a,b]$ satisfying $(*)$

(*1) implies $Au=f$ where $A=-\partial_x^2+\mathbb{1}$ operator.

idea:
① 放大空间 $X \supset C'$ ← 增加对象
② 试找方程成立的弱化 (weak top) ← 降低方程 ← 方程容易满足

$u \in C([a,b]) \subseteq$ unknown function space $X$ ← 放大搜索网
宁可错杀一千, 不能一人漏网

try to find a candidate of solution in this larger space $X$

A powerful tool to enlarge the function space is the dual space.

roughly speaking, the dual space of a relatively "small" space is larger. From the functional analysis (distribution theory)

the dual of a small function space $C_0^\infty$, $(C_0^\infty)^*$ could includ $C([a,b])$.

Take $C([a,b]) \subseteq X \subseteq (C_0^\infty)^*$ (AKA. $C_c^\infty$)

topology is taken by the weak top. (Coarsest top) of $(C_0^\infty)^*$

i.e. make all $\varphi \in C_0^\infty$, $\langle \varphi, f \rangle$ are continuous i.e. make all $\varphi \in C_0^\infty$, $\langle \varphi, f \rangle$ are continuous

$\uparrow$ $C([a,b])$

$\Updownarrow$

$\left( f_n \xrightarrow{\text{weak}} f \iff \langle \varphi, f_n \rangle \to \langle \varphi, f \rangle, \text{ as } n \to \infty, \text{ for all } \varphi \in C_0^\infty \right)$

Remark: Step 1: Find weak solutions. ← complete space $X$ is important

Step 2: Existence and Uniqueness (compactness in certain top. then iterations; completions; simplify the solution.)

Step 3: regularities to conclude weak is classical.

to materialize step 1, we introduce Sobolev spaces.

NEED $X$
1. $C^1 \subset X \subset (C_0^\infty)^*$
2. with weak top associating with $(C_0^\infty)^*$
3. complete space under its own strong top.
4. ... other properties such as regularity for integrability ← include info. on frequency (regularity)

In $X$ with weak topology associating with $(C_0^\infty)^*$, $(*)$ becomes

$$-\int_a^b u'' \varphi + \int_a^b u \varphi = \int_a^b f \varphi \quad \forall \varphi \in C_0^1([a,b]), \quad \varphi(a) = \varphi(b) = 0$$

AKA test function.

↑ why this is eq in weak top??

the space of infinitely differentiable functions with compact supp in $[a,b]$

$Au = f$

$Au = f$

limiting info e.g. $A = -\partial^2 + \mathbb{1}$

derivatives ← lim process ← Strong lim in strong top.

need to replace this strong top. to weak top.

$Au = f$ ← strong limit

$(Au)_n \xrightarrow{\text{strong}} Au$ replaced by $(Au)_n \xrightarrow{\text{weak}} Au$

i.e. $\langle \varphi, (Au)_n \rangle \longrightarrow \langle \varphi, Au \rangle \; \forall \varphi \in C_0^\infty$

$\langle \varphi, Au \rangle = \langle \varphi, f \rangle$ when $\langle f, g \rangle = \int_a^b f g \, dx$.

Note that

$$\int_a^b u'' \varphi \, dx = -\int_a^b u' \varphi' \, dx \quad \text{integration by parts}$$

then the weak formulation becomes.

$$\int_a^b u' \varphi' \, dx + \int_a^b u \varphi \, dx = \int_a^b f \varphi \, dx$$

☆ all the limiting process can be replaced by weak topology-related limit

only need $u \in C^1$ instead of $u \in C^2$ previously.

From this, we give a def. of weak derivative.

Def: Suppose $u, v \in L^1_{loc}(U)$ and $\alpha$ is a multi-index, we say that $v$ is the $\alpha$th-weak partial derivative of $u$, denoted by $D^\alpha u = v$

provided

$$\boxed{\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx}$$

motivated by integration by parts.

for all test functions $\phi \in C_0^\infty(U)$.

for all test functions $\phi \in C_0^\infty(U)$.

related to the derivative of distributions

$\langle D^\alpha u, \phi \rangle = \langle u, D^\alpha \phi \rangle (-1)^{|\alpha|} \quad \forall \phi \in \mathcal{D}(\Omega)$ Schwartz

How to understand this def.

let $|\alpha| = 1$ then $\langle Du, \phi \rangle = -\langle u, D\phi \rangle$

In fact, this is the real derivative in weak top.

Redefine $\tau_{-h}\varphi(x) = \varphi(x-h)$

then $\dfrac{\varphi(x-h) - \varphi(x)}{h} = \dfrac{\tau_{-h}\varphi - \varphi}{h}$

let us see, $\lim\limits_{h \to 0} \dfrac{\tau_{-h}u - u}{h}$, if this does not exist. the lim-related topology is too strong, you can't find the limit. However, if using weak topology, the limit could exist. i.e. try to find

$\dfrac{\tau_{-h}u - u}{h} \xrightarrow{\text{weak}} ???$

$\Leftrightarrow \left\langle \varphi, \dfrac{\tau_{-h}u - u}{h} \right\rangle \to \langle \varphi, ??? \rangle \quad \forall \varphi \in C_0^\infty$

$||$ (under the left side): $\dfrac{1}{h}\left( \langle \varphi, \tau_{-h}u \rangle - \langle \varphi, u \rangle \right)$

$|||$ (under $???$): $-v$

$||$

$\langle \varphi, \tau_{-h}u \rangle - \langle \varphi(x), u(x-h) \rangle$

$$\frac{1}{h}\left(\langle \tau_h\varphi, u\rangle - \langle \varphi, u\rangle\right)$$

$$\| $$

$$\left\langle \frac{\tau_h\varphi-\varphi}{h}, u\right\rangle$$

$\langle \varphi, \tau_{-h}u\rangle = \langle \varphi(x), u(x-h)\rangle$
$= \int \varphi(x)\, u(x-h)\, dx$
$= \int \varphi(y+h) u(y)\, dy = \langle \varphi(y+h), u(y)\rangle$
$= \langle \tau_h\varphi, u\rangle$

i.e. $\left\langle \varphi, \frac{\tau_{-h}u-u}{h}\right\rangle = \left\langle \frac{\tau_h\varphi-\varphi}{h}, u\right\rangle$

Then try to find $v$ s.t. $\lim_{h\to 0}\left\langle \varphi, \frac{\tau_{-h}u-u}{h}\right\rangle = -\langle \varphi, v\rangle$

which becomes $\lim_{h\to 0}\left\langle \frac{\tau_h\varphi-\varphi}{h}, u\right\rangle = -\langle \varphi, v\rangle$

i.e. $\boxed{\langle D\varphi, u\rangle = -\langle \varphi, v\rangle}$

i.e. $\boxed{\langle \varphi, v\rangle = \lim_{h\to 0}\left\langle \varphi, \frac{u(x)-u(x-h)}{h}\right\rangle} = \lim_{h\to 0}\left\langle \varphi, \frac{u(x+h)-u(x)}{h}\right\rangle$

Comparing with the strong derivative,

$\exists\, v$ s.t. $v = \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$ denote $v = Du$

Lemma (Uniqueness of weak derivatives). A weak $\alpha^{th}$-partial derivative of $u$, if it exists, is uniquely defined up to a set of measure zero.

Pf: Assume that $v, \tilde{v} \in L^1_{loc}(U)$ satisfy

Pf Assume that $v, \tilde{v} \in L^1_{loc}(U)$ satisfy

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi \, dx$$

for all $\phi \in C_0^\infty(U)$. Then

$$\int_U (v - \tilde{v}) \phi \, dx = 0 \quad \text{for all } \phi \in C_0^\infty(U)$$

$\Rightarrow v - \tilde{v} = 0$ a.e. $\square$

known as Fundamental lemma of the calculus of variations



An alternative of the function space $X$ is

$\boxed{\text{Sobolev spaces}}$.

Fix $1 \le p \le \infty$ and let $k \ge 0$ integer

**Def**: The Sobolev space $W^{k,p}(U)$

consists of all locally summable ($L^1_{loc}$) functions $u : U \to \mathbb{R}$ such that for each multiindex $\alpha$ with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$

i.e. $W^{k,p}(U) := \{ u \in L^1_{loc}(U) : D^\alpha u \in L^p \text{ for all } |\alpha| \le k \}$

weak derivatives

weak derivatives

Remark: If $p=2$. denote $H^k(U) = W^{k,2}(U)$

since this $H^k$ is a Hilbert space. and $H^0(U)=L^2(U)$

☆ Remark: Dual weak formulations can eliminate the spatial derivatives effectively to simplify the equation in most cases, and sometimes the equation can be turned to a ODE in some sense.

One can differentiate the equations many times and express it in dual weak formulation. at every level of the (weak) differentiation of the eq. one can study it in $L^p$ spaces. That's why we abstract the definition of Sobolev spaces in such way.

Another point of view from Fourier analysis. in which one can see why the frequency is in which one can see why the frequency is important. we neglect this at this moment.

**Def:** (norms of $W^{k,p}(U)$) strong topology which is easy to use.

If $u \in W^{k,p}(U)$, we define its norm to be

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left( \sum_{|\alpha| \le k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}} & (1 \le p < \infty) \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty) \end{cases}$$

**Remark:** 不唯一，很多等价 norms. e.g. $\|u\|_{W^{k,p}} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p}$

**Def:** (1) Let $\{u_m\}_{m=1}^{\infty}$, $u \in W^{k,p}$, We say

$u_m \to u$ in $W^{k,p}(U)$

$\iff \lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(U)} = 0$

(2) $u_m \to u$ in $W^{k,p}_{loc}(U)$

$\iff u_m \to u$ in $W^{k,p}_{loc}(V)$ for every $V \subset\subset U$.

means $\bar{V} \subset U$ & $\bar{V}$ compact
$V$ strictly contained in $U$

**Def:** $W^{k,p}_0(U) = \overline{C_0^\infty(U)}^{W^{k,p}(U)}$ closure by using topology

**Def:** $W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{W^{k,p}(U)}$ closure by using topology of $W^{k,p}(U)$, $\|\cdot\|_{W^{k,p}}$

**Remarks:**

① This implies that $\boxed{u \in W_0^{k,p}(U)}$

$\Updownarrow$

$\exists$ a series of function $u_m \in C_c^\infty(U)$ s.t.

$u_m \to u$ in $W^{k,p}(U)$ $\left( \|u_m - u\|_{W^{k,p}} \to 0 \right)$

② In fact, $W_0^{k,p}(U) = \{ u \in W^{k,p}(U) : D^\alpha u = 0 \text{ on } \partial U, \ \forall |\alpha| \le k-1 \}$

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## Elementary properties

1. Properties of weak derivatives. These properties obvious hold for smooth functions. Sobolev functions may not be smooth.

**Thm 1:** Assume $u, v \in W^{k,p}(U)$, $|\alpha| \le k$, then

(1) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all $\alpha, \beta$ with $|\alpha| + |\beta| \le k$.

(2) (linearity) $\forall \lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \le k$

(3) If $V$ is an open subset of $U$, then $u \in W^{k,p}(V)$

(4) If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

(4) If $\zeta \in C_0^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

$$D^\alpha(\zeta u) = \sum_{|\beta| \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta \, D^{\alpha-\beta} u \quad \text{(Leibniz's rule)}$$

where $\binom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$.

Pf: (1) $\forall \varphi \in C_0^\infty(U)$
$$\langle D^\beta(D^\alpha u), \varphi \rangle = (-1)^{|\beta|} \langle D^\alpha u, D^\beta \varphi \rangle = (-1)^{|\beta|+|\alpha|} \langle u, D^\alpha D^\beta \varphi \rangle$$
$$= (-1)^{|\beta|+|\alpha|} \langle u, D^{\alpha+\beta} \varphi \rangle = \langle D^{\alpha+\beta} u, \varphi \rangle$$

$$\Rightarrow D^\beta(D^\alpha u) = D^{\alpha+\beta} u.$$

(2) $\langle \cdot, \cdot \rangle$ linear $\Rightarrow$ conclusion.

(3) By definition directly.

(4) Can be proved by induction. only see $|\beta|=|\alpha|=1$, i.e. we want to prove
$$D(\zeta u) = D\zeta \cdot u + \zeta Du$$

$\forall \varphi \in C_0^\infty$

$$\langle D(\zeta u), \varphi \rangle = \int_U D(\zeta u) \varphi \, dx = -\int_U \zeta u \, D\varphi \, dx$$

weak derivative

$\because \zeta \in C_0^\infty$
$$= -\int u \cdot D(\zeta\varphi) + \int u \, D\zeta \cdot \varphi$$

$C_0^\infty$

weak derivative

$$= \underbrace{\int Du \cdot \xi \varphi}_{} + \int u D\xi \cdot \varphi$$

↓ weak derivative $C_0^\infty$

$$= \int (Du \cdot \xi + u D\xi)\varphi$$

$$\Leftrightarrow \quad D(\xi u) = Du \cdot \xi + u D\xi \qquad \square$$

**Thm 2.** (Completeness) Sobolev space $W^{k,p}(U)$ ($k = 1, 2, \infty$, $p \in [1, \infty]$) is a Banach space $\begin{cases} \text{normed space} \\ \text{completeness} \end{cases}$

**Pf:** ① Check $\|u\|_{W^{k,p}(U)}$ is a norm by def.

② Completeness: Assume $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $W^{k,p}(U)$

i.e. $u_m \in W^{k,p}$ for all $m = 1, \cdots, \infty$.

Then $D^\alpha u_m \in L^p$ for all $|\alpha| \le k$.

$\because L^p$ complete $\Rightarrow \exists\, u_\alpha \in L^p$ s.t. $D^\alpha u_m \to u_\alpha$ in $L^p(U)$

and $u_m \to u$ in $L^p(U)$

③ Prove $u_\alpha = D^\alpha u$

where $u_\alpha$, $u$ defed as above.

to achieve this, by def. of weak der., only require that $\forall \varphi \in C_0^\infty(U)$

$$\Big[\ \langle u_\alpha, \varphi \rangle \overset{\text{strong lim} \Rightarrow \text{weak convergence}}{=} \lim \langle D^\alpha u_m, \varphi \rangle \overset{\text{def of weak deriv.}}{=} (-1)^{|\alpha|} \lim \langle u_m, D^\alpha \varphi \rangle$$

$$\begin{cases} \langle u_\alpha, \varphi \rangle \overset{\text{strong lim} \Rightarrow \text{weak convergence}}{=} \lim\limits_{m\to\infty} \langle D^\alpha u_m, \varphi \rangle \overset{\text{def of weak deriv.}}{=} (-1)^{|\alpha|} \lim\limits_{m\to\infty} \langle u_m, D^\alpha \varphi \rangle \\ \langle u, D^\alpha \varphi \rangle \overset{\text{Strong} \Rightarrow \text{weak}}{=} \lim\limits_{n\to\infty} \langle u_m, D^\alpha \varphi \rangle \end{cases}$$

加负号

$$\Rightarrow \langle u_\alpha, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle \qquad \forall \varphi \in C_0^\infty(U)$$

Then $u_\alpha = D^\alpha u$. $\square$.

# Approximations & Mollifiers

Wednesday, January 1, 2020 10:29 AM

This section is a useful tool for studying profound properties of Sobolev space

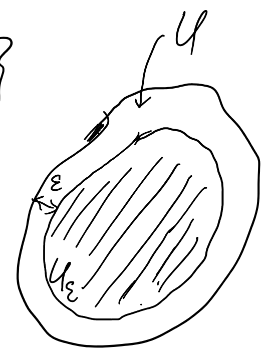
Motivation: develop some systematic procedures for approximating a $W^{k,p}$ function by smooth functions

Then any inequality of Sobolev space can be reduced to the inequalities of smooth functions (by approximating to derive the ineq. in Sobolev spaces)

$\boxed{\text{A tool of the tool : Mollifier}}$

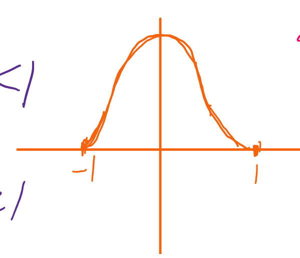
A notation: If $U \subset \mathbb{R}^n$ is open and $\varepsilon > 0$, we denote

$$U_\varepsilon := \{ x \in U : \operatorname{dist}(x, \partial U) > \varepsilon \}$$




Def: (1) Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$




suitable bump function

$\quad 0 \qquad \text{if } |x| \geq 1$

the constant $C>0$ selected so that $\int_{\mathbb{R}^n} \eta \, dx = 1$

(2) For each $\varepsilon > 0$, set

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right) \leftarrow \text{宽度变成}[-\varepsilon, \varepsilon]\text{，体积仍保持1.}$$

we call $\eta$ the standard mollifier. The function $\eta_\varepsilon$ are $C^\infty$ and satisfy $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$, $\operatorname{supp}(\eta_\varepsilon) \subset B(0, \varepsilon)$.

**Def**: If $f: U \to \mathbb{R}$ is locally integrable, define its mollification $f^\varepsilon := \eta_\varepsilon * f$ in $U_\varepsilon$ 内部 interior.

That is,

$x \in U_\varepsilon$，对 $x \in U \setminus U_\varepsilon$, then $x-y$ may be in the exterior of $U_\varepsilon$. $x-y \notin U_\varepsilon$

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x-y) f(y) \, dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y) \, dy, \text{ for } x \in U_\varepsilon.$$

Convolution: modify the shape of $f$ by $\eta_\varepsilon(x-y)$，以$x$为中心按mollifier的分布来加权平均$f$，这样修改$f$的分布.

在Fourier freq. space中，$f$是特定freq，$\eta$对应的freq，按$\eta$，$x$中心分布分离中特定区域的freq 加权平均给$f^\varepsilon(x)$.

i.e. $f^\varepsilon(x)$函数的每一点$x$的值是$x$周围按$\eta$分布分离出来的freq $f$的加权平均.

We can prove some nice properties of mollifiers which is powerful in analysis

**Thm**: (Properties of mollifiers). (omit the pf)

Thm: (Properties of mollifiers). (omit the pf)

(1) $f^\varepsilon \in C^\infty(U_\varepsilon)$ ← "locally integrable" becomes "smooth" (mollifier)

(2) $f^\varepsilon \to f$ a.e. as $\varepsilon \to 0$. ← smooth modifications tends to locally integ. function $f$ "pointwisely"

$f^\varepsilon = \eta_\varepsilon * f$

(3) If $f \in C(U)$, then $f^\varepsilon \to f$ uniformly (in this norm) on compact subsets of $U$.

If $f$ is better: "continuous" instead of "locally integrable", then this convergence is "uniformly"

uniformly means $\sup_{x\in U} |f^\varepsilon - f| \to 0$ as $\varepsilon \to 0$

"uniform norm = sup norm of $C(U)$"

(4) If $1 \le p < \infty$ and $f \in L^p_{loc}(U)$, then $f^\varepsilon \to f$ in $L^p_{loc}(U)$ (in this norm)

☆ exclude $\infty$

$(\eta_\varepsilon * f \to f)$

## Interior approximation by smooth functions

Thm 1. (local approximation by smooth functions). Assume $u \in W^{k,p}(U)$ for some $1 \le p < \infty$, and set $u^\varepsilon = \eta_\varepsilon * u$ in $U_\varepsilon$

Then (1) $u^\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$, ← by above Thm (1) and

(2) $u^\varepsilon \to u$ in $W^{k,p}_{loc}(U)$, as $\varepsilon \to 0$.

☆ $\Rightarrow$ $\boxed{\overline{C_0^\infty(\mathbb{R}^n)}^{W^{k,p}(\mathbb{R}^n)} = W^{k,p}(\mathbb{R}^n)}$ see Adam Corollary 3.23 $W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$

(i.e. $C_0^\infty(\mathbb{R}^n)$ is dense $W^{k,p}(\mathbb{R}^n)$) $= W_0^{k,p}(\mathbb{R}^n)$

证: only need to prove (2), to achieve this, we claim:

Small lemma: 若 $|\alpha| \le k$, then

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \quad \text{in } U_\varepsilon \qquad \left(\text{i.e. } \boxed{D^\alpha(\eta_\varepsilon * u) = \eta_\varepsilon * D^\alpha u}\right)$$

weak derivative (since we want to use $W^{k,p}_{loc}(U)$)

Pf of lemma:

$D^\alpha u^\varepsilon(x) = D^\alpha[\eta_\varepsilon * u(x)] = D^\alpha \int \eta_\varepsilon(x-y)\, u(y)\, dy$

$$D^\alpha u^\varepsilon(x) = D^\alpha\left[\eta_\varepsilon * u(x)\right] = D_x^\alpha \int_U \eta_\varepsilon(x-y)\, u(y)\, dy$$

$$= \int_U D_x^\alpha\left[\eta_\varepsilon(x-y)\, u(y)\right] dy = \int_U D_x^\alpha \eta_\varepsilon(x-y)\, u(y)\, dy$$

$D_x \eta_\varepsilon(\overset{\xi}{\overset{\shortparallel}{x-y}}) = D_\xi \eta_\varepsilon \cdot \frac{\partial \xi}{\partial x} = D_\xi \eta_\varepsilon$

$D_y \eta_\varepsilon(x-y) = D_\xi \eta_\varepsilon \cdot \frac{\partial \xi}{\partial y} = -D_\xi \eta_\varepsilon$

$D_x$ change to $D_y$

$$= (-1)^{|\alpha|} \int_U D_y^\alpha \underbrace{\eta_\varepsilon(x-y)}_{\substack{\Uparrow \\ C_0^\infty}} u(y)\, dy$$

$C_0^\infty$ by its construction.

$$= \underbrace{(-1)^{2|\alpha|}}_{=1} \int_U \eta_\varepsilon(x-y)\, D_y^\alpha u(y)\, dy = \eta_\varepsilon * D^\alpha u. \quad \#$$

Then for any open set $V \subset\subset U$, By the property (4) of mollifiers, we conclude that $D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \to D^\alpha u$ in $L^p(V)$, since $D^\alpha u \in L^p(V)$ ($\because u \in W^{k,p}(U)$), as $\varepsilon \searrow 0$, $\forall |\alpha| \le k$.

Therefore, $\|u^\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\le k} \|D^\alpha u^\varepsilon - D^\alpha u\|^p_{L^p(V)} \to 0$. as $\varepsilon \to 0$. #

**Approximation by smooth functions**

no need to read for the first readers

see §5.3.2. and §5.3.3.

if assume $U$ is bounded. (or even better $U$ is bounded & $\partial U$ is $C^1$)

$\{u_m\}$ can be better, see $u_m \in C^\infty(U) \cap W^{k,p}(U)$

(or even better $u_m \in C^\infty(\bar{U})$)

5.4 Extensions give you a way to extend functions in $W^{1,p}(U)$ to functions in $W^{1,p}(\mathbb{R}^n)$

5.5 Trace give boundary values.

5.5 Trace give boundary values.

# Sobolev embedding thms. (inequalities)

Wednesday, January 1, 2020 4:48 PM

As we mentioned before, Uncertainty Principle provides a constraint on the "height" and "frequency": $A \gtrsim \frac{1}{\sqrt{Nd}}$, this is the Sobolev embedding thm (or Sobolev-type inequalities) which trades regularities for integrabilities

higher integrability means the accuracy of the function.

If $U$ is bounded, then $L^1 \supseteq L^2 \supseteq \cdots \supseteq L^p \supseteq \cdots \supseteq L^\infty$

[Thm: (An Imbedding thm for $L^p$ space (see Adam Thm 2.14, P28))

Suppose that $vol(\Omega) = \int_\Omega 1dx < \infty$ and $1 \le p \le q \le \infty$. If $u \in L^q(\Omega)$ then $u \in L^p(\Omega)$ and $\|u\|_{L^p} \le (vol(\Omega))^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^q}$

Hence $L^q(\Omega) \hookrightarrow L^p(\Omega)$.

Pf: Hölder inequality. $\int_\Omega |u(x)|^p dx \le \left(\int_\Omega |u(x)|^q dx\right)^{\frac{p}{q}} \left(\int_\Omega 1 dx\right)^{1-\frac{p}{q}}$ #.]

Focus on simple case $W^{1,p}(U)$ and ask:

Question: If a function $u \in W^{1,p}(U)$, then does $u$ automatically belong to certain other spaces? Yes!

depends on $p$
$$\begin{cases} 1 \le p < n & \text{☆ focus} \\ p = n & \\ n < p \le \infty & \end{cases}$$
$\}$ ← mention them (for $p=n$ and $n<p\le\infty$)

$\boxed{\text{Gagliardo-Nirenberg-Sobolev inequality}}$ ← Assume $1 \le p < n$

The goal of this section is an estimate

(*) $\|u\|_{L^q(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)} \ldots$

$$(*) \quad \|u\|_{L^q(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)}$$

for certain constants $C>0$, $1\le q<\infty$ and all function $u\in C_c^\infty(\mathbb{R}^n)$

independ of $u$.

The idea of the proof: ① prove this inequality for smooth function.

② Using the approximations in above lecture. i.e. smooth function are dense in Sobolev space.

③ we only care about $\mathbb{R}^n$. for a subset $U\subset\mathbb{R}^n$, one can apply extensions.

$\boxed{\text{A necessary condition on } q}$

Consistent with Uncertainty Principle

Claim: We first point out that $q$ cannot be arbitrary by scaling property

Pf: $\forall u\in C_c^\infty(\mathbb{R}^n)$, $u\ne 0$. define, for $\lambda>0$, the rescaled function

$$u_\lambda(x) := u(\lambda x) \qquad (x\in\mathbb{R}^n)$$

Apply $(*)$ to $u_\lambda$, we derive that

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C\|Du_\lambda\|_{L^p(\mathbb{R}^n)} \quad (**)$$

Note that

① $\|u_\lambda\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|u_\lambda|^q dx = \int_{\mathbb{R}^n}|u(\underset{y}{\lambda x})|^q dx = \int_{\mathbb{R}^n}\frac{1}{\lambda^n}|u(y)|^q dy$

② $\|Du_\lambda\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|Du_\lambda|^p dx = \int_{\mathbb{R}^n}|D_x u(\underset{y}{\lambda x})|^p dx$

$$= \frac{1}{\lambda^n}\int_{\mathbb{R}^n} |D_x u(y)|^p \, dy = \frac{\lambda^p}{\lambda^n}\int_{\mathbb{R}^n} |D_y u(y)|^p \, dy = \frac{\lambda^p}{\lambda^n}\int_{\mathbb{R}^n} |Du(y)|^p \, dy$$

insert to (**)

$$\Rightarrow \frac{1}{\lambda^{\frac{n}{q}}}\left(\int_{\mathbb{R}^n} |u(y)|^q \, dy\right)^{\frac{1}{q}} \le C\, \frac{\lambda}{\lambda^{\frac{n}{p}}}\left(\int_{\mathbb{R}^n} |Du(y)|^p \, dy\right)^{\frac{1}{p}}$$

$$\Rightarrow \left(\int_{\mathbb{R}^n} |u(y)|^q \, dy\right)^{\frac{1}{q}} \le C\, \lambda^{1-\frac{n}{p}+\frac{n}{q}} \left(\int_{\mathbb{R}^n} |Du(y)|^p \, dy\right)^{\frac{1}{p}}$$

$$\Rightarrow \|u\|_{L^q(\mathbb{R}^n)} \le C\, \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$\Rightarrow 1-\frac{n}{p}+\frac{n}{q} = 0$, otherwise the coef's depend on $\lambda$, then it could be $0$ or $\infty$.

$$\Rightarrow \boxed{q = \frac{np}{n-p}.} \qquad \#$$

useful! then make it to be a definition.

**Def.** If $1 \le p < n$, the Sobolev conjugate of $p$ is

$$p^* := \frac{np}{n-p}.$$

Note $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, $p^* > p$

After knowing such necessary condition, we next propose the G-N-S inequality

G-N-S inequality

Thm (Gagliardo-Nirenberg-Sobolev inequality)

Assume $1 \le p < n$. There exists a constant $C$, depending only on $p$ and $n$, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_c^1(\mathbb{R}^n)$ ← Remark: this can be generalized to $\forall u \in W^{1,p}(\mathbb{R}^n)$

need compact support. note $u \equiv 1$ ✗

Pf: Step 1: $p = 1$. $p^* = \frac{n}{n-1}$ Aim: $\|u\|_{L^{\frac{n}{n-1}}} \le C\|Du\|_{L^1}$

$\forall u \in C_c^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $i = 1, \cdots, n$

↓ integrable

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)\, dy_i$$

eventually it's Fundamental thm of calculus.

$$\Rightarrow |u(x)| \le \int_{-\infty}^{\infty} |Du(x_1, \cdots, y_i, \cdots x_n)|\, dy_i \quad (\text{for any } i = 1, \cdots, n)$$

$$\Rightarrow |u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} |Du(x_1, \cdots, y_i, \cdots x_n)|\, dy_i \right)^{\frac{1}{n-1}}$$

$$\Rightarrow \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 \le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1$$

$f_i$ and $p_i = \frac{1}{n-1} \Rightarrow \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$

$$= \left( \int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \boxed{\left( \int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}}}\, dx_1$$

$\|\prod_{i}^{n} f_i\| \le \prod^{m} \|f_i\|$

$$= \left(\int_{-\infty} \underbrace{|Du| dy_1}\right) \int_{-\infty} \prod_{i=2} \left(\int_{-\infty} |Du| dy_i\right)^{n-1} dx_1$$

$\|\prod_{i=1}^{n} f_i\|_{L^1} \le \prod_{i=1}^{m} \|f_i\|_{L^{\frac{1}{n-1}}}$

变量 $Du(y_1, x_2, \cdots, x_n)$

变量为 $(x_2, \cdots, x_n)$

General Hölder ineq

$1 \le p_1, \cdots, p_m \le \infty$, $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{r}$. $u_k \in L^{p_k}$, then

$$\|\prod_{k=1}^{m} u_k\|_{L^r} \le \prod_{k=1}^{m} \|u_k\|_{L^{p_k}(U)}$$

$$I_1(x_2, \cdots, x_n)$$

$$\le \underbrace{\left(\int_{-\infty}^{\infty} |Du| dy_1\right)^{\frac{1}{n-1}}} \prod_{i=2}^{n} \underbrace{\left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du| dy_i\right) dx_1\right]}_{I_i}^{\frac{1}{n-1}} \quad (*)$$

Integrate $(*)$ w.r.t $x_2$

e.g. $I_2(x_3, \cdots, x_n)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2 \le \int_{-\infty}^{\infty} \left(I_1^{\frac{1}{n-1}} \prod_{i=2}^{n} I_i^{\frac{1}{n-1}}\right) dx_2$$

$$= I_2^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{\substack{i=1 \\ i \ne 2}}^{n} I_i^{\frac{1}{n-1}} dx_2$$

If $n=2$. $\int_{-\infty}^{\infty} I_1^{\frac{1}{n-1}} I_2^{\frac{1}{n-1}} dx_2 = I_2^{\frac{1}{n-1}} \int_{-\infty}^{\infty} I_1^{\frac{1}{n-1}} dx_2$ end of story

$$= \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_2 dx_1\right)^{\frac{1}{n-1}} \underbrace{\int_{-\infty}^{\infty} \prod_{\substack{i=1 \\ i \ne 2}}^{n} I_i^{\frac{1}{n-1}} dx_2}$$

$\le$ Hölder again

$$\left(\int_{-\infty}^{\infty} I_1 dx_2\right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} I_i dx_2\right)^{\frac{1}{n-1}}$$

$$\le \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_2 dx_1\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_1 dx_2\right)^{\frac{1}{n-1}}$$

$$\prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dx_2 dy_i\right)^{\frac{1}{n-1}}$$

Continue to integrate w.r.t $x_3, \cdots, x_n$,

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dx_1 \cdots dy_i \cdots dx_n\right)^{\frac{1}{n-1}}$$

$$\int_{\mathbb{R}^n} \cdots \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \, dy_1 \cdots dy_n \right)$$

$$(**) \qquad = \left( \int_{\mathbb{R}^n} |Du| \, dx \right)^{\frac{n}{n-1}} \qquad \#$$

**Step 2:** $1 < p < n$. let $v := |u|^{\gamma}$ $(\gamma > 1)$ and using $(**)$, $v \in C_0^{\infty}$

$$\left( \int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} \gamma |u|^{\gamma - 1} |Du| \, dx$$

$$|D(|u|^m)| = |m|u|^{m-1} D|u|| = \left| \frac{u}{|u|} \cdot m|u|^{m-1} \right| \cancel{m u |u|^{m-2} Du} = m|u|^{m-1}|Du|$$

$$\overset{\text{Hölder}}{\le} \gamma \left( \int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx \right)^{1-\frac{1}{p}} \underbrace{\left( \int_{\mathbb{R}^n} |Du|^p dx \right)^{\frac{1}{p}}}_{= \|Du\|_{L^p}}$$

Hoping they can be cancelled.

by choosing $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} \Rightarrow \gamma = \frac{p(n-1)}{n-p} > 1.$

$$\Rightarrow \frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$$

then $\left( \int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\frac{n-1}{n}} \le \gamma \left( \int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{1-\frac{1}{p}} \|Du\|_{L^p}$

$$\Rightarrow \left( \int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\overbrace{1-\frac{1}{n}-1+\frac{1}{p}}^{=\frac{1}{p^*}}} \le \gamma \|Du\|_{L^p}$$

$$\Rightarrow \|u\|_{L^{p^*}} \le \gamma \|Du\|_{L^p}. \qquad \#$$

**Pf of Remark** (ie. $\forall u \in W^{1,p}(\mathbb{R}^n)$)

Since $\overline{C_0^{\infty}(\mathbb{R}^n)}^{W^{1,p}(\mathbb{R}^n)} = W^{1,p}(\mathbb{R}^n)$ ($C_0^{\infty}(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$)

there exists a sequence $u_m \in C_0^{\infty}(\mathbb{R}^n)$ s.t.

there exists a sequence $u_j \in C_0^\infty(\mathbb{R}^n)$ s.t.

$u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$ ($\|Du_j - Du\|_{L^p} \le \|u_j - u\|_{L^p} + \|Du_j - Du\|_{L^p} \to 0$)

By above thm. we have $\|u_j - u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du_j - Du\|_{L^p(\mathbb{R}^n)}$

$\to 0$ as $j \to \infty$.

$\Rightarrow u_j \to u$ in $L^{p^*}$ $\because$ norm $\|\cdot\|$ is a continuous function.

$$\Rightarrow \left.\begin{array}{ccc} \|u_j\|_{L^{p^*}} & \le & C\|Du_j\|_{L^p} \\ \downarrow j\to\infty & & \downarrow j\to\infty \\ \|u\|_{L^{p^*}} & & \|Du\|_{L^p} \end{array}\right\} \Rightarrow \|u\|_{L^{p^*}} \le C\|Du\|_{L^p}$$

for any $u \in W^{1,p}(\mathbb{R}^n)$ #

Remark: ① Thm 2. states that If $U$ bded. open & $\partial U \in C^1$ then this thm still holds for $\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)}$.

and $u \in W^{1,p}(U)$ $\Leftarrow$ $C_0^\infty$ dense in $W^{1,p}$. by approximation.

② Thm 3. State that $u \in W_0^{1,p}(U)$

$\boxed{\text{Morrey's inequality}}$ $n < p \le \infty$ case $\Rightarrow$ $W^{1,p}(\mathbb{R}^n) \subset C^{0,\gamma}(\mathbb{R}^n)$ (Continuous embedding)

where $\gamma := 1 - \frac{n}{p}$.

Continuous embedding: Let $X, Y$ be two normed vector spaces with norms $\|\cdot\|_X$ & $\|\cdot\|_Y$ respectively, such that $X \subseteq Y$. If the inclusion (identity) map: $i: X \hookrightarrow Y: x \mapsto x$ is continuous, i.e. if there exists a constant $C$, s.t. $\|x\|_Y \le C\|x\|_X$ for every $x \in X$, then $X$ is called continuously embedded in $Y$.

Compact embedding: Let $X, Y, \ldots$ ——— with norms $\|\cdot\|_X$ & $\|\cdot\|_Y$ respectively

if ① $X$ is continuously embedded in $Y$

② The embedding of $X$ into $Y$ is a compact operator

any bounded set in $X$ is precompact in $Y$

precompact $A \Rightarrow \bar{A}$ compact.

# General Sobolev inequalities

Generalize above thms

Thm (General Sobolev ineqs) Let $U$ be a bounded open subset of $\mathbb{R}^n$, with a $C^1$ boundary. Assume $u \in W^{k,p}(U)$.

(1) If $k < \frac{n}{p}$, then $u \in L^q(U)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.

$\boxed{W^{k,p}(U) \subset L^p}$

Continuous embedding

we have in addition the estimate $\|u\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)}$

the constant $C$ depending only on $k, p, n$ and $U$.

(2) If $k > \frac{n}{p}$ then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})$, where

$\boxed{W^{k,p}(U) \subset C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})}$

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$$\gamma = \begin{cases} [\frac{n}{p}]+1-\frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} <1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})} \leq C\|u\|_{W^{k,p}(U)},$$

the constant $C$ depending only on $k, p, n, \gamma$ and $U$.

Pf: (1) only Prove (1) $\because k<\frac{n}{p}$. then $D^\alpha u \in L^p(U)$ for all $|\alpha| \leq k$, we can use G-N-S inequality on $D^\beta u$.

若 $U=\mathbb{R}^n$. 若 $U$ bdd $\partial\Omega \in C^1$ $\leq C\|D^\beta u\|_{W^{1,p}(U)}$

$$\|D^\beta u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|DD^\beta u\|_{L^p(\mathbb{R}^n)} \quad \text{for } \forall |\beta|+1 \leq k$$

for general case.

$$\Rightarrow \|D^\beta u\|_{L^{p^*}(U)} \leq C\|D^\beta u\|_{W^{1,p}(U)} \leq C\|u\|_{W^{k,p}(U)}$$

Sum all $\beta$ up $\Rightarrow$ $\boxed{\|u\|_{W^{k-1,p^*}(U)} \leq C\|u\|_{W^{k,p}(U)}}$ $(*)$

then $\xRightarrow{\text{repeatedly using } (*)}$ $\|u\|_{W^{k-2,p^{**}}(U)} \leq C\|u\|_{W^{k-1,p^*}} \leq C\|u\|_{W^{k,p}(U)}$

$$\Rightarrow \underbrace{\|u\|_{W^{k-k,p^{\overbrace{*\cdots*}^{k}}}(U)}}_{\| \atop L^{p^{\overbrace{*\cdots*}^{k}}} =: L^q} \leq \cdots \leq \|u\|_{W^{k-2,p^{**}}(U)} \leq \cdots \leq C\|u\|_{W^{k,p}(U)}$$

$\boxed{\text{finite times}}$

$$\Rightarrow \|u\|_{L^q} \leq C\|u\|_{W^{k,p}(U)}.$$

and $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}$, $\frac{1}{p^{**}}=\frac{1}{p^*}-\frac{1}{n}=\frac{1}{p}-\frac{2}{n}$

$\cdots, \frac{1}{q} = \frac{1}{p^{*\cdots *}} - \frac{1}{n} = \frac{1}{p} - \underbrace{\frac{1}{n} \cdots - \frac{1}{n}}_{k} = \frac{1}{p} - \frac{k}{n}$ #

## $\boxed{\text{Compact embeddings}}$

We have known the continuous embedding $W^{1,p} \hookrightarrow L^{p^*}(U)$ for $1 \le p < n$ & $p^* = \frac{np}{n-p}$. Now we demonstrate that $W^{1,p}(U)$ is in fact compactly embedded in $L^q(U)$ for $1 \le q < p^*$.

motivation: $X \subset\subset Y$, if we can derive a sequence $x_n \in X$ is bded, then $\{x_n\}$ have a limit in $Y$, usually this $\lim_{n\to\infty} x_n$ is the "solution" of the PDE. "Boundness" is easier to prove than "Compactness", so this compact embedding has similar use as Weierstrass thm in calculus

Thm: (Rellich-Kondrachov Compactness Thm, Sobolev compact embedding)

Assume $U$ is a bounded open subset of $\mathbb{R}^n$ and $\partial U$ is $C^1$. Suppose $1 \le p < n$. Then

$$W^{1,p}(U) \subset\subset L^q(U) \quad \text{for each } 1 \le q < p^*.$$