

Emergence of nonlinear Jeans-type instabilities for quasilinear wave equations

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§1. Introduction

(Equations, goals, motivations and main results)

Equations and goals

$$\partial_t^2 \varrho - \left(\frac{m^2 (\partial_t \varrho)^2}{(1 + \varrho)^2} + 4(k - m^2)(1 + \varrho) \right) \Delta \varrho = F(t, \varrho, \partial_\mu \varrho)$$

where the nonlinear source terms are

$$\begin{aligned} F(t, \varrho, \partial_\mu \varrho) := & \underbrace{\frac{2}{3} \varrho(1 + \varrho)}_{\text{(i) self-increasing}} \underbrace{- \frac{1}{3} \partial_t \varrho}_{\text{(ii) damping}} + \underbrace{\frac{4}{3} \frac{(\partial_t \varrho)^2}{1 + \varrho}}_{\text{(iii) Riccati}} \\ & + \underbrace{\left(m^2 \frac{(\partial_t \varrho)^2}{(1 + \varrho)^2} + 4(k - m^2)(1 + \varrho) \right) q^i \partial_i \varrho - K^{ij}(t, \varrho, \partial_\mu \varrho) \partial_i \varrho \partial_j \varrho}_{\text{(iv) convection}}. \end{aligned}$$

$$\varrho|_{t=t_0} = \underbrace{\beta}_{\text{const.}} + \underbrace{\psi(x^k)}_{\text{inhomogeneity}} \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \underbrace{\beta_0}_{\text{const.}} + \underbrace{\psi_0(x^k)}_{\text{inhomogeneity}}, \quad \text{in } \{t_0\} \times \mathbb{R}^n,$$

- (Goal) Find **self-increasing blowup** solutions (formations of nonlinear cosmological structures).
- (Result) The solutions can attain **arbitrarily large values** over time, leading to **self-increasing singularities at some future endpoints of null geodesics**, provided the **inhomogeneous perturbations of data** are sufficiently **small** (**long wave feature!**).

After **time transform** $t \rightarrow \ln t$, the equation becomes:

$$\begin{aligned} \partial_t^2 \varrho - g^{ij} \partial_i \partial_j \varrho &= \frac{2}{3t^2} \varrho(1 + \varrho) - \frac{4}{3t} \partial_t \varrho + \frac{4}{3} \frac{(\partial_t \varrho)^2}{1 + \varrho} + g q^i \partial_i \varrho \\ &\quad - \frac{1}{t^2} K^{ij}(t, \varrho, \partial_\mu \varrho) \partial_i \varrho \partial_j \varrho, \quad \text{in } [t_0, t^*) \times \mathbb{R}^n, \\ \varrho|_{t=t_0} &= \beta + \psi(x^k) \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \beta_0 + \psi_0(x^k), \quad \text{in } \{t_0\} \times \mathbb{R}^n, \end{aligned}$$

where

$$g^{ij} = g^{ij}(t, \varrho, \partial_t \varrho) := g(t, \varrho, \partial_t \varrho) \delta^{ij} = \left(m^2 \frac{(\partial_t \varrho)^2}{(1 + \varrho)^2} + 4(k - m^2) \frac{1 + \varrho}{t^2} \right) \delta^{ij}.$$

- Now **focus on this equation!!** A time transform $t \rightarrow e^t$ leads back to the previous equation.

A short version for the motivation

Relate to the famous problem in astrophysics:

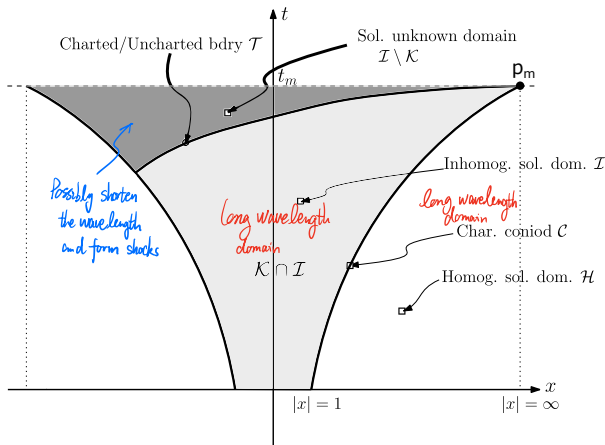
Important astrophysical question

- Cosmological perturbation theories (linear, higher order approximations) by e.g., Jeans, Bonner, Lifshitz, Bardeen, Hawking, Ellis, Bruni, Zel'dovich, Peebles, Brandenberger, Mukhanov, Hwang, Noh... (long lists in astrophy). Stability/instability;
 - (Jeans instability) The formations of the nonlinear cosmological structures;
 - (Core mathematical mechanism) Nonlinear Jeans-type instability. Rare mathematical studies currently.
-
- Our final goal is the nonlinear Jeans instability for the Euler–Poisson and the Einstein–Euler system.
 - A toy model for above system. Neglecting rotations, shears of the fluids, and tidal forces, Euler–Poisson (or Einstein–Euler) leads to this type of QNLW.

Assumptions for simplifications

- $m^2 \leq k$, $\beta \in (0, +\infty)$, $\beta_0 \in (0, +\infty)$, $t_0 \in (0, +\infty)$. $\psi \in C_0^1(\mathbb{R}^n)$ and $\psi_0 \in C_0^1(\mathbb{R}^n)$ are given **positive-valued** functions with $\text{supp}\psi = B_1(0)$ and $\text{supp}\psi_0 = B_1(0)$. K_{ij} be analytic functions in all their variables. The initial data satisfies $\beta_0^2 \geq 4\beta(1 + \beta)^2 t_0^{-2}$;
- The **direction of convection** is assumed to be **constant** and can be normalized $q^i = |q|\delta_1^i$ and $|q| \in (3, 100)$ (**intro. a directional bias**);
- $k = \frac{1}{4}$.

Main Theorem



$$\mathcal{C} := \left\{ (t, x) \in [t_0, t_m) \times \mathbb{R}^n \mid |x| = 1 + \int_{t_0}^t \sqrt{g(y, f(y), f_0(y))} dy \right\}.$$

Remark (directional bias)

The direction of convection is assumed to be constant and can be normalized $q^i = |q| \delta_1^i$.

Main Theorem

Suppose $k \in \mathbb{Z}_{\frac{n}{2}+3}$, A, B, C, D are constants depending on the initial data β and β_0 , as defined in the article, and that Assumptions hold. Let $(\psi, \psi_0) \in C_0^1(\mathbb{R}^n)$ be given functions with $\text{supp}(\psi, \psi_0) = B_1(0)$, $f(t)$ be the solution to the key reference ODE (later!). Then there exist sufficiently small constants $\sigma_0 > 0$ and $\delta_0 > 0$, such that if the initial data satisfy

$$\|\psi\|_{H^k(B_1(0))} + \|\partial_i \psi\|_{H^k(B_1(0))} + \|\psi_0\|_{H^k(B_1(0))} \leq e^{-\frac{153}{\delta_0}} \sigma_0^2,$$

then there exists a hypersurface $t = \mathcal{T}(x, \delta_0)$ satisfying

$$\Gamma_{\delta_0} := \{(t, x) \in [t_0, t_m) \times \mathbb{R}^n \mid t = \mathcal{T}(x, \delta_0)\} \subset \mathcal{I}, \quad \lim_{a \rightarrow +\infty} \mathcal{T}(a\delta_1^i, \delta_0) = t_m$$

$$\lim_{\delta_0 \rightarrow 0+} \mathcal{T}(x, \delta_0) = t_m,$$

such that there is a solution $\varrho \in C^2(\mathcal{K} \cup \mathcal{H})$ to the main equation where $\mathcal{K} := \{(t, x) \in [t_0, t_m) \times \mathbb{R}^n \mid t < \mathcal{T}(x, \delta_0)\}$ satisfying

Main Theorem (conti.)

- if we denote

$$\mathbf{1}_-(x^1) := 1 - C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} \searrow 1 \quad \text{and} \quad \mathbf{1}_+(x^1) := 1 + C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} \searrow 1, \quad \text{as } x^1 \rightarrow +\infty,$$

then there are estimates for $(t, x) \in \mathcal{K} \cap \mathcal{I}$,

$$\begin{aligned} \mathbf{1}_-(x^1)f_0(t_0 + \mathbf{1}_-(x^1)(t - t_0)) &\leq \varrho_0(t, x) \leq \mathbf{1}_+(x^1)f_0(t_0 + \mathbf{1}_+(x^1)(t - t_0)), \\ -C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} (1 + f(t_0 + \mathbf{1}_-(x^1)(t - t_0))) &\leq \varrho_i(t, x) \leq C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} (1 + f(t_0 + \mathbf{1}_+(x^1)(t - t_0))), \\ \mathbf{1}_-(x^1)f(t_0 + \mathbf{1}_-(x^1)(t - t_0)) &\leq \varrho(t, x) \leq \mathbf{1}_+(x^1)f(t_0 + \mathbf{1}_+(x^1)(t - t_0)). \end{aligned}$$

Moreover, both ϱ_0 and ϱ reach the self-increasing singularities at the point $p_m := (t_m, +\infty, 0, \dots, 0)$:

$$\begin{aligned} \lim_{\mathcal{K} \ni (t, x) \rightarrow p_m} \varrho &= \lim_{\mathcal{K} \ni (t, x) \rightarrow p_m} f = +\infty, \\ \lim_{\mathcal{K} \ni (t, x) \rightarrow p_m} \varrho_0 &= \lim_{\mathcal{K} \ni (t, x) \rightarrow p_m} f_0 = +\infty \quad \text{and} \quad \lim_{\mathcal{K} \ni (t, x) \rightarrow p_m} \varrho_i = 0. \end{aligned}$$

- $\varrho \equiv f$ for $(t, x) \in \mathcal{H}$.

Main Theorem (conti.)

- the growth rate of ϱ can be estimated using visualizable functions

$$\varrho(t, x) \geq \mathbf{1}_-(x^1) f(t_0 + \mathbf{1}_-(x^1)(t - t_0)) > \mathbf{1}_-(x^1) (e^{C(t_0 + \mathbf{1}_-(x^1)(t - t_0))^{\frac{2}{3}}} - 1)$$

and

$$\varrho(t, x) \leq \mathbf{1}_+(x^1) f(t_0 + \mathbf{1}_+(x^1)(t - t_0)) < \frac{3}{2} \left(\frac{1}{1 + \frac{A}{t_0 + \mathbf{1}_+(x^1)(t - t_0)} + B(t_0 + \mathbf{1}_+(x^1)(t - t_0))^{\frac{2}{3}}} - 1 \right)$$

for all $(t, x) \in \mathcal{K} \cap \mathcal{I}$.

- if the initial data satisfy $\check{\beta} := \frac{t_0 \beta_0}{1 + \beta} - 1 > 0$, then ϱ has an improved lower bound, indicating finite-time blowups.

$$\varrho(t, x) \geq \mathbf{1}_-(x^1) f(t_0 + \mathbf{1}_-(x^1)(t - t_0)) > \mathbf{1}_-(x^1) \left(\frac{1 + \beta}{\left(\frac{\beta_0 t_0^{\frac{4}{3}}}{1 + \beta} (t_0 + \mathbf{1}_-(x^1)(t - t_0))^{-\frac{1}{3}} - \check{\beta} \right)^3} - 1 \right)$$

for all $(t, x^k) \in \mathcal{K} \cap \mathcal{I}$.

Generations

remark

The convection term introduces a directional bias in gq^i , causing the wave to propagate more strongly in a particular direction depending on the sign and magnitude of gq^i .

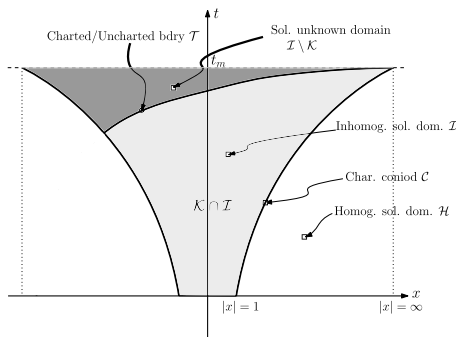


Figure: $q^i = |q|\delta_1^i$

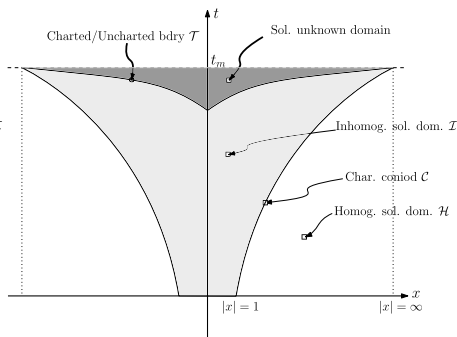


Figure: $q^i \propto x^i/|x|$

*§2. Backgrounds: classical Jeans instabilities
(only include classical ones without further developments in astrophysics.)*

Classical Jeans instability (Static, Euler–Poisson system)

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_i (\rho v^i) = 0 \\ \partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0 \\ \delta^{ij} \partial_i \partial_j \phi = 4\pi G \rho \end{array} \right.$$

Euler – Poisson system

let $\rho = \underbrace{\rho_0}_{\substack{\uparrow \\ \text{constant}}} + \tilde{\rho}$, $v^i = \underbrace{v_0}_{\substack{\uparrow \\ 0}} + \tilde{v}^i$, $\phi = \underbrace{\phi_0}_{\substack{\uparrow \\ \nabla \phi_0 = 0 \text{ contradicts Poisson.}}} + \tilde{\phi}$, $\tilde{p} = c_s^2 \tilde{\rho}$

① linearization
 $\Rightarrow$

$$\left\{ \begin{array}{l} \partial_t \tilde{\rho} + \rho_0 \partial_i \tilde{v}^i = 0 \\ \partial_t \tilde{v}^i + \frac{c_s^2}{\rho_0} \partial^i \tilde{\rho} + \partial^i \tilde{\phi} = 0 \quad \xrightarrow{\oplus \partial_i} \\ \delta^{ij} \partial_i \partial_j \tilde{\phi} = 4\pi G \tilde{\rho} \end{array} \right. \quad \begin{array}{l} \square \tilde{\rho} - k^2 \tilde{\rho} = 0, (k > 0) \\ \boxed{\partial_t^2 \tilde{\rho} - c_s^2 \delta^{ij} \partial_i \partial_j \tilde{\rho} - 4\pi G \rho_0 \tilde{\rho} = 0} \\ \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\ \quad \quad \text{Const.} \quad \quad \text{Const.} \end{array}$$

$$\partial_t^2 \tilde{\rho} - c_s^2 \delta_{ij} \partial_i \partial_j \tilde{\rho} - 4\pi G \rho_0 \tilde{\rho} = 0$$

Fourier transform.

$$\tilde{\rho}_k'' + (k^2 c_s^2 - 4\pi G \rho_0) \tilde{\rho}_k = 0 \Rightarrow \begin{cases} \tilde{\rho}_k \propto \exp(\pm i \omega(k) t) \\ \omega(k) = \sqrt{k^2 c_s^2 - 4\pi G \rho_0} \end{cases}$$

if $k^2 c_s^2 - 4\pi G \rho_0 < 0 \Leftrightarrow k < \frac{\sqrt{4\pi G \rho_0}}{c_s} =: k_J$

$$\Rightarrow \tilde{\rho}_k \propto \exp(\pm | \omega | t)$$

exponentially growth.

Classical Jeans instability (expansion)

Expanding Newtonian Universe

$$\left\{ \begin{aligned} \rho &= \rho_0(t) = \frac{1}{6\pi G t^2} \quad , \quad v^i = v_0^i = \frac{2}{3t} x^i \quad (H(t) = \frac{2}{3t} \text{ Hubble's law}) \\ \phi &= \phi_0 = \frac{2}{3} \pi G \rho_0 |x|^2 \end{aligned} \right.$$

let $p = p_0 + \tilde{p}$, $v_i = v_{i0} + \tilde{v}_i$, $\phi = \phi_0 + \tilde{\phi}$

$$H = \frac{\dot{a}}{a}$$

density contrast

① Lagrangian coord. (comoving with Hubble flow) : $x^i = a(t) q^i$

② Linearization Euler-Poisson

③ ∂_i (momentum conservation) ← Poisson eq.
continuity eq.

let $\epsilon = \frac{\tilde{\rho}}{\rho_0}$

$\Rightarrow \partial_t^2 \epsilon + 2H \partial_t \epsilon - \frac{c_s^2}{a^2} \partial_i \partial_i \epsilon - 4\pi G \rho_0 \epsilon = 0$

$$\partial_t^2 e + \frac{4}{3t} \dot{e} - \frac{c_s^2}{a^2} \delta^{ij} \partial_i \partial_j e - \frac{2}{3t^2} e = 0$$

$\Downarrow$ Fourier transform

$$e_k'' + \frac{4}{3t} e_k' + \left(\frac{c_s^2 k^2}{a^2} - \frac{2}{3t^2} \right) e_k = 0.$$

$\Downarrow$ c_s small (pressure small)

$$e_k'' + \frac{4}{3t} e_k' - \frac{2}{3t^2} e_k = 0$$

$\Downarrow$ Euler ODE

$$e_k = C_1 t^{-1} + C_2 t^{\frac{2}{3}} \Rightarrow |e| \sim t^{\frac{2}{3}}.$$

§3. Our previous works on Jeans instabilities

Our previous works on nonlinear Jeans instability

- (Extremely simplified nonlinear model) Rigorous proof of slightly nonlinear Jeans instability in the expanding Newtonian universe. Physical Review D (PRD), 2022, 105(4): 043519.
- (Mathematical model) Blowups for a class of second order nonlinear hyperbolic equations: A reduced model of nonlinear Jeans instability. arXiv:2208.06788.
- (New exact Jeans instable solution to Euler-Poisson) Fully nonlinear gravitational instabilities for expanding Newtonian universes with inhomogeneous pressure and entropy: Beyond the Tolman's solution. Physical Review D (PRD), 2023, 107(12): 123534.
- (Nonlinear Jeans instability for Euler-Poisson with specific source) Fully nonlinear gravitational instabilities for expanding spherical symmetric Newtonian universes with inhomogeneous density and pressure. arXiv:2305.13211.

Try 1: Simple nonlinearity and Fuchsian formulations

$$\ddot{\varrho} + \frac{4}{3t}\dot{\varrho} - \tilde{\kappa}t^{-2\gamma+\frac{2}{3}}\Delta\varrho - \frac{2}{3t^2}\varrho = (\gamma-1)\tilde{\kappa}t^{-2\gamma+\frac{2}{3}}\frac{D^i\varrho D_i\varrho}{1+\varrho}. \quad (1)$$

Theorem

Suppose $s \in \mathbb{Z}_{\geq 3}$ and $\gamma > 1$ are constants and $\dot{\varrho} := \varrho|_{t=1}$ and $\dot{\varrho}_\mu := (\partial_\mu \varrho)|_{t=1}$ ($\mu = 0, \dots, 3$). Let the initial data of the density satisfies an estimate

$$\left\| \dot{\varrho} - \frac{\beta}{2} \right\|_{H^s(\mathbb{T}^3)} + \left\| \dot{\varrho}_0 - \frac{\beta}{3} \right\|_{H^s(\mathbb{T}^3)} + \|\dot{\varrho}_i\|_{H^s(\mathbb{T}^3)} \leq \beta_0, \quad (2)$$

where $0 < \beta < +\infty$ is any given constant and $\beta_0 > 0$ is a small enough constant. Then the solution of equation (1) satisfies

$$\frac{1}{4}\beta t^{\frac{2}{3}} \leq \varrho \leq \frac{3}{4}\beta t^{\frac{2}{3}}$$

for every $(t, \mathbf{q})$.

- Non-Fourier based method: **Fuchsian formulations** (energy method)
- Main difficulties:
 - ① Find a **compactified time** $\tau \in [-1, 0)$ for physical time $t \in [t_0, \infty)$ such that $\tau = g(t)$.
 - ② Select proper **Fuchsian fields** (similar define suitable energies)

Tool: Fuchsian formulations (KEY TOOL 1)

$$B^\mu \partial_\mu u = \frac{1}{t} \mathbf{B} \mathbf{P} u + G \quad \text{in } [-1, 0) \times \mathbb{T}^n,$$
$$u = u_0 \quad \text{on } \{-1\} \times \mathbb{T}^n.$$

Some main assumptions of this system

- ① $\mathbf{P}$ is a constant, symmetric projection operator (Pick up decay variables by $\mathbf{P}$).
- ② $\frac{1}{\gamma_1} \mathbb{I} \leq \dot{\mathbf{B}}^0 \leq \frac{1}{\kappa} \dot{\mathbf{B}} \leq \gamma_2 \mathbb{I}$ (Give right signs and determine dissipative effects).
- ③ $(B^\mu)^T = B^\mu$, $[\mathbf{P}, \mathbf{B}] = \mathbf{P}\mathbf{B} - \mathbf{B}\mathbf{P} = 0$.
- ④ $\mathbf{P}^\perp B^0(t, \mathbf{P}^\perp u) \mathbf{P} = \mathbf{P} B^0(t, \mathbf{P}^\perp u) \mathbf{P}^\perp = 0$.
- ⑤ $|\mathbf{P}^\perp [D_u B^0(t, u) (B^0)^{-1} \mathbf{B} \mathbf{P} u] \mathbf{P}^\perp|_{op} \leq \alpha |t| + \beta |\mathbf{P} u|^2$.
- ⑥ e.g., B^i , G somehow allows $\sim 1/t$ (extra condition) and $\sim 1/\sqrt{t}$

(3, 4, 5 gives how the variables coupling to each other), and some regularity assumptions on the coefficients and remainders. Advantage: allow suitable coupling of variables.

Theorem (Oliynyk, 2016, Pfs by energy methods)

Suppose that $k \geq \frac{n}{2} + 1$, $u_0 \in H^k(\mathbb{T}^n)$ and above conditions are fulfilled. Then there exists a $T_* \in (-1, 0)$, and a unique classical solution $u \in C^1([-1, T_*] \times \mathbb{T}^n)$ that satisfies $u \in C^0([-1, T_*], H^k) \cap C^1([-1, T_*], H^{k-1})$ and the energy estimate

$$\|u(t)\|_{H^k}^2 - \int_{-1}^t \frac{1}{\tau} \|\mathbf{P}u\|_{H^k}^2 d\tau \leq C e^{C(t+1)} (\|u(-1)\|_{H^k}^2)$$

for all $-1 \leq t < T_*$, where $C = C(\|u\|_{L^\infty([-1, T_*], H^k)}, \gamma_1, \gamma_2, \kappa)$, and can be uniquely continued to a larger time interval $[T_0, T^*)$ for all $T^* \in (T_*, 0]$ provided $\|u\|_{L^\infty([-1, T_*], W^{1,\infty})} < \infty$.

This basic theorem has been generalized to more difficult cases and two parameter scales problems in the subsequent works by Oliynyk, Beyer, Olvera-Santamaría.

Intuitive toy model of Cauchy problem for Fuchsian system

- Rough idea: The following ODE dominated behaviors.
Consider an ODE

$$\partial_t u = \frac{\beta}{t} u + (-t)^{-1+p} F(t), \quad \text{where } 0 < p \leq 1, \beta > 0, t \in [-1, 0).$$

Then

$$\partial_t \left(u - \int_{-1}^t \frac{\beta}{s} u ds \right) = (-t)^{-1+p} F(t).$$

Integrating it yields

$$u - \int_{-1}^t \frac{\beta}{s} u ds \lesssim u_0 + 1 - (-t)^p.$$

Further solving u leads to optimal decay estimates.

- The previous Theorem is obtained **by adding conditions to make sure the Fuchsian system behaves like this toy model.**

Try 2: Composite nonlinearities but with synchronizable sources

$$\square \varrho(x^\mu) + \frac{a}{t} \partial_t \varrho(x^\mu) - \frac{b}{t^2} \varrho(x^\mu)(1 + \varrho(x^\mu)) - \frac{c - k}{1 + \varrho(x^\mu)} (\partial_t \varrho(x^\mu))^2 = kF(t),$$

$$\varrho|_{t=t_0} = \varrho_0(x^i) > 0 \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \varrho_0'(x^i) > 0,$$

where $\square := \partial_t^2 - \Delta_g = \partial_t^2 - g^{ij}(t) \partial_i \partial_j$,

$$a > 1, \quad b > 0, \quad 1 < c < 3/2$$

$$g^{ij}(t) := \frac{m^2 (\partial_t f(t))^2}{(1 + f(t))^2} \delta^{ij} \quad \text{and} \quad F(t) := \frac{(\partial_t f(t))^2}{1 + f(t)},$$

where $m \in \mathbb{R}$ is a given constant and $f(t)$ solves an ODE,

$$\partial_t^2 f(t) + \frac{a}{t} \partial_t f(t) - \frac{b}{t^2} f(t)(1 + f(t)) - \frac{c}{1 + f(t)} (\partial_t f(t))^2 = 0,$$

$$f(t_0) = \beta > 0 \quad \text{and} \quad \partial_t f(t_0) = \beta_0 > 0.$$

The solutions of ODEs (KEY TOOL 2)

$$\partial_t^2 f(t) + \frac{a}{t} \partial_t f(t) - \frac{b}{t^2} f(t)(1 + f(t)) - \frac{c}{1 + f(t)} (\partial_t f(t))^2 = 0,$$
$$f(t_0) = \beta > 0 \quad \text{and} \quad \partial_t f(t_0) = \beta_0 > 0.$$

Theorem

- ① $t_\star \in [0, \infty)$ exists and $t_\star > t_0$;
- ② (**Blowups**) there is a constant $t_m \in [t_\star, \infty]$, such that there is a unique solution $f \in C^2([t_0, t_m))$ to the ODE, and

$$\lim_{t \rightarrow t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow t_m} f_0(t) = +\infty.$$

- ③ (**Estimates of growth rates of f**) f satisfies upper and lower bound estimates,

$$1 + f(t) > \exp\left(Ct^{\frac{\bar{a}+\Delta}{2}} + Dt^{-1}\right) \quad \text{for } t \in (t_0, t_m);$$

$$1 + f(t) < \left(At^{\frac{\bar{a}-\Delta}{2}} + Bt^{\frac{\bar{a}+\Delta}{2}} + 1\right)^{-1} \quad \text{for } t \in (t_0, t_\star).$$

$$\partial_t^2 f(t) + \frac{a}{t} \partial_t f(t) - \frac{b}{t^2} f(t)(1 + f(t)) - \frac{c}{1 + f(t)} (\partial_t f(t))^2 = 0,$$

$$f(t_0) = \beta > 0 \quad \text{and} \quad \partial_t f(t_0) = \beta_0 > 0.$$

Theorem

Furthermore, if the **initial data satisfies** $\beta_0 > \bar{a}(1 + \beta)/(\bar{c}t_0)$, then

- ④ t_* and t^* exist and finite, and $t_0 < t_* < t^* < \infty$;
- ⑤ there is a finite time $t_m \in [t_*, t^*)$, such that there is a solution $f \in C^2([t_0, t_m])$ to the ODE, and

$$\lim_{t \rightarrow t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow t_m} f_0(t) = +\infty.$$

- ⑥ (**Improved lower bounds, finite time blowups**) the solution f has improved lower bound estimates, for $t \in (t_0, t_m)$,

$$(1 + \beta)(1 - Et_0^{\bar{a}} + Et^{\bar{a}})^{1/\bar{c}} < 1 + f(t).$$

Pf: A series of **differential inequalities** in the form of $\partial_t G < (>) \frac{k}{t} G + F$.

The solutions to the PDEs

$$\square \varrho(x^\mu) + \frac{a}{t} \partial_t \varrho(x^\mu) - \frac{b}{t^2} \varrho(x^\mu) (1 + \varrho(x^\mu)) - \frac{c - k}{1 + \varrho(x^\mu)} (\partial_t \varrho(x^\mu))^2 = kF(t),$$

$$\varrho|_{t=t_0} = \varrho_0(x^i) > 0 \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \varrho_0'(x^i) > 0,$$

- **Conclusions:** ϱ has self-increasing singularities at $t = t_m$ and has the growth rates $\sim f$.

The compactified time (KEY TOOL 3)

- Methods: Fuchsian formulations.
- The compactified time

$$\begin{aligned}\tau := -g(t) &= -\exp\left(-A \int_{t_0}^t \frac{f(s)(f(s)+1)}{s^2 f_0(s)} ds\right) \\ &= -\left(1 + bB \int_{t_0}^t s^{a-2} f(s)(1+f(s))^{1-c} ds\right)^{-\frac{A}{b}} \in [-1, 0).\end{aligned}$$

Fuchsian fields

$$w(t, x^i) := \varrho(t, x^i) - f(t),$$

$$w_0(t, x^i) := \partial_t w(t, x^i) = \partial_t \varrho(t, x^i) - f_0(t),$$

$$w_i(t, x^i) := \partial_i w(t, x^i) = \partial_i \varrho(t, x^i).$$

and

$$u(t, x^i) = \frac{1}{f(t)} w(t, x^i),$$

$$u_0(t, x^i) = \frac{1}{f_0(t)} w_0(t, x^i),$$

$$u_i(t, x^i) = \frac{m}{1 + f(t)} w_i(t, x^i).$$

then

$$\underline{u}(\tau, x^i) = u(g^{-1}(-\tau), x^i), \quad \underline{u}_0(\tau, x^i) = u_0(g^{-1}(-\tau), x^i)$$

$$\underline{u}_i(\tau, x^i) = u_i(g^{-1}(-\tau), x^i).$$

Singular/Regular- τ terms w/ special quantities and hidden identities of f (KEY TOOL 4)

Define a quantity

$$\chi(t) := \frac{t^{2-a}f_0(t)}{(1+f(t))^{2-c}f(t)g^{\frac{b}{A}}(t)} = \frac{g^{-\frac{2b}{A}}(t)t^{2(1-a)}}{Bf(t)(1+f(t))^{2(1-c)}}.$$

Then there is a function $\mathfrak{G} \in C^1([t_0, t_m))$, such that for $t \in [t_0, t_m)$,

$$\chi(t) = \frac{2bB}{3-2c} + \mathfrak{G}(t).$$

where $\lim_{t \rightarrow t_m} \mathfrak{G}(t) = 0$. Moreover, there is a constant $C_\chi > 0$ such that $0 < \chi(t) \leq C_\chi$ in $[t_0, t_m)$, and there are continuous extensions of χ and $\mathfrak{G}$ such that $\chi \in C^0([t_0, t_m])$ and $\mathfrak{G} \in C^0([t_0, t_m])$ by letting $\chi(t_m) := 2bB/(3-2c)$ and $\mathfrak{G}(t_m) := 0$.

Define a quantity

$$\xi(t) := 1/[g(t)(1 + f(t))],$$

then $\xi \in C^1([t_0, t_m))$ and

$$\lim_{t \rightarrow t_m} \xi(t) = 0.$$

Moreover, there is a constant $C_\star > 0$, such that $0 < \xi(t) \leq C_\star$ for every $t \in [t_0, t_m)$, and there is a continuous extension of ξ such that $\xi \in C^0([t_0, t_m])$ by letting $\xi(t_m) := 0$.

Remark

$\chi(t)$ and $\xi(t)$ help distinguish the singular term $\frac{1}{\tau} \mathbf{B} \mathbf{P} u$ and the regular term G in the Fuchsian system:

$$B^\mu \partial_\mu u = \frac{1}{\tau} \mathbf{B} \mathbf{P} u + G.$$

Try 3&4: Nonlinear gravitational instabilities (applications)

The dimensionless and normalized Euler–Poisson system

$$\begin{aligned}\partial_t \rho + \partial_i(\rho v^i) &= 0, \\ \partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi &= 0, \\ \partial_t s + v^i \partial_i s &= 0, \\ \Delta \phi &= \delta^{ij} \partial_i \partial_j \phi = 4\pi \rho.\end{aligned}$$

The *equation of state* becomes

$$p = Ke^s \rho^{\frac{4}{3}} + \mathfrak{p}, \quad \text{for } K \geq 0.$$

There is an exact solution on $(t, x^k) \in [t_0, \infty) \times \mathbb{R}^3$,

$$\begin{aligned}\dot{\rho}(t) &= \frac{\iota^3}{6\pi t^2}, \quad \dot{p}(t) = K t^{-\frac{4}{3}} \delta_{kl} x^k x^l \dot{\rho}^{\frac{4}{3}} + \mathfrak{p}, \quad \dot{v}^i(t, x^k) = \frac{2}{3t} x^i, \\ \dot{\phi}(t, x^k) &= \frac{2}{3} \pi \dot{\rho} \delta_{ij} x^i x^j = \frac{\iota^3}{9t^2} \delta_{ij} x^i x^j, \quad \dot{s}(t, x^k) = \ln(t^{-\frac{4}{3}} \delta_{kl} x^k x^l)^{\text{sgn}(1-\iota^3)},\end{aligned}$$

Method: **ODE from Key tool 2.**

We construct solutions

$$\begin{aligned}\rho(t) &= (1 + f(t))\dot{\rho}(t) = \frac{\iota^3(1 + f(t))}{6\pi t^2}, \\ v^i(t, x^i) &= \frac{2}{3t}x^i - \frac{f'(t)}{3(1 + f(t))}x^i, \\ \phi(t, x^i) &= \frac{2}{3}\pi\dot{\rho}(1 + f(t))|\mathbf{x}|^2 = \frac{\iota^3(1 + f(t))|\mathbf{x}|^2}{9t^2}, \\ s(t, x^k) &= \ln\left(t^{-\frac{4}{3}}(1 + f)^{\frac{2}{3}}\delta_{kl}x^kx^l\right)^{\text{sgn}(1-\iota^3)}.\end{aligned}$$

and the *density contrast* $\varrho(t) = f(t)$ where $|\mathbf{x}|^2 := \delta_{ij}x^ix^j$ and $f(t)$ is a solution of the following nonlinear ODE,

$$\begin{aligned}f''(t) + \frac{4}{3t}f'(t) - \frac{2}{3t^2}f(t)(1 + f(t)) - \frac{4(f'(t))^2}{3(1 + f(t))} &= 0, \\ f|_{t=t_0} = \beta \quad \text{and} \quad f'|_{t=t_0} &= 3(1 + \beta)\gamma.\end{aligned}$$

Moreover, the pressure becomes $p(t) = \frac{K\iota^4}{(6\pi)^{\frac{4}{3}}t^4}(1 + f)^2\delta_{kl}x^kx^l$.

- **Result: Self-increasing singularities.**

Try 3&4: Nonlinear gravitational instabilities (applications)

The dimensionless and normalized Euler–Poisson system

$$\begin{aligned}\partial_t \rho + \partial_i (\rho v^i) &= 0, \\ \partial_t v^i + v^j \partial_j v^i + \frac{\partial^i \rho}{\rho} + \partial^i \phi &= \mathcal{D}^i(t, x^j, \rho, v^k, s, \phi), \\ \partial_t s + v^i \partial_i s &= \mathcal{S}(t, x^j, \rho, v^k, s, \phi), \\ \Delta \phi &= \delta^{ij} \partial_i \partial_j \phi = 4\pi \rho.\end{aligned}$$

EoS is

$$p = K e^s \rho^{\frac{4}{3}} \quad \text{for } K > 0.$$

- $\mathcal{S}$ and $\mathcal{D}$ provide the **synchronizable source** like F .
- Transform to a type of **Try 2**;
- **Self-increasing singularities**.

Eventually, we arrive at

$$\square_g \hat{\varrho} + \left(\frac{4}{3t} + \frac{\kappa f_0}{1+f} \right) \partial_t \hat{\varrho} - \frac{2}{3t^2} \hat{\varrho}(1 + \hat{\varrho}) - \frac{4(\partial_t \hat{\varrho})^2}{3(1 + \hat{\varrho})} = F_1, \quad \text{Target eq.}$$

$$\partial_t \nu + \frac{f_0}{3(1+f)} \nu \partial_\zeta \nu = G_1, \quad \text{Transport eq.,}$$

where the wave operator is

$$\square_g := \partial_t^2 - g^{\zeta\zeta} \partial_\zeta^2 + 2g^{0\zeta} \partial_\zeta \partial_t,$$

$$g^{\zeta\zeta} := \frac{(2+\omega)(1-\iota^3)}{9t^2} \frac{(1+\hat{\varrho})^{\omega+1}}{(1+f)^\omega} - \frac{f_0^2}{9(1+f)^2} \nu^2, \quad g^{0\zeta} := \frac{f_0}{3(1+f)} \nu.$$

Summary of tools

- Fuchsian GIVP;
- The reference ODE of f ;
- Hidden quantities and identities of f (distinguishing **Singular** and **Regular- τ** terms);
- Time compactifications.

§4. Emergence of nonlinear Jean-type instabilities for QNLW
1. Main Theorems (review)

Equations and goals

$$\partial_t^2 \varrho - \left(\frac{m^2 (\partial_t \varrho)^2}{(1 + \varrho)^2} + 4(k - m^2)(1 + \varrho) \right) \Delta \varrho = F(t, \varrho, \partial_\mu \varrho)$$

where the nonlinear source terms are

$$\begin{aligned} F(t, \varrho, \partial_\mu \varrho) := & \underbrace{\frac{2}{3} \varrho(1 + \varrho)}_{\text{(i) self-increasing}} \underbrace{- \frac{1}{3} \partial_t \varrho}_{\text{(ii) damping}} + \underbrace{\frac{4}{3} \frac{(\partial_t \varrho)^2}{1 + \varrho}}_{\text{(iii) Riccati}} \\ & + \underbrace{\left(m^2 \frac{(\partial_t \varrho)^2}{(1 + \varrho)^2} + 4(k - m^2)(1 + \varrho) \right) q^i \partial_i \varrho - K^{ij}(t, \varrho, \partial_\mu \varrho) \partial_i \varrho \partial_j \varrho}_{\text{(iv) convection}}. \end{aligned}$$

$$\varrho|_{t=t_0} = \underbrace{\beta}_{\text{const.}} + \underbrace{\psi(x^k)}_{\text{inhomogeneity}} \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \underbrace{\beta_0}_{\text{const.}} + \underbrace{\psi_0(x^k)}_{\text{inhomogeneity}}, \quad \text{in } \{t_0\} \times \mathbb{R}^n,$$

- (Goal) Find **self-increasing blowup** solutions (formations of nonlinear cosmological structures).
- (Result) The solutions can attain **arbitrarily large values** over time, leading to **self-increasing singularities at some future endpoints of null geodesics**, provided the **inhomogeneous perturbations of data** are sufficiently **small** (**long wave feature!**).

After **time transform** $t \rightarrow \ln t$, the equation becomes:

$$\begin{aligned} \partial_t^2 \varrho - g^{ij} \partial_i \partial_j \varrho &= \frac{2}{3t^2} \varrho(1 + \varrho) - \frac{4}{3t} \partial_t \varrho + \frac{4}{3} \frac{(\partial_t \varrho)^2}{1 + \varrho} + g q^i \partial_i \varrho \\ &\quad - \frac{1}{t^2} K^{ij}(t, \varrho, \partial_\mu \varrho) \partial_i \varrho \partial_j \varrho, \quad \text{in } [t_0, t^*) \times \mathbb{R}^n, \\ \varrho|_{t=t_0} &= \beta + \psi(x^k) \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \beta_0 + \psi_0(x^k), \quad \text{in } \{t_0\} \times \mathbb{R}^n, \end{aligned}$$

where

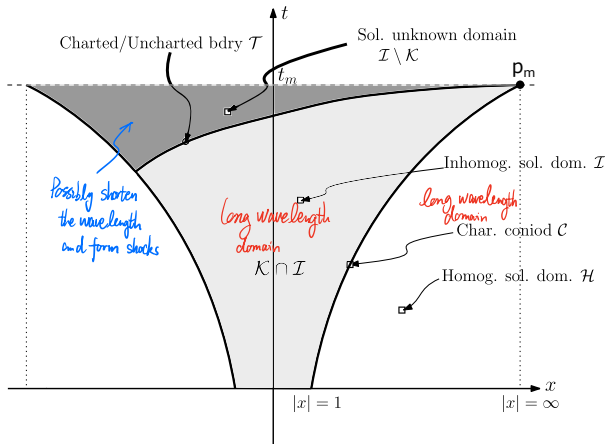
$$g^{ij} = g^{ij}(t, \varrho, \partial_t \varrho) := g(t, \varrho, \partial_t \varrho) \delta^{ij} = \left(m^2 \frac{(\partial_t \varrho)^2}{(1 + \varrho)^2} + 4(k - m^2) \frac{1 + \varrho}{t^2} \right) \delta^{ij}.$$

Remark

Self-increasing/Riccati terms **(dominant)** $\sim \frac{4}{3} \frac{(\partial_t f)^2}{1 + f} \sim \frac{2}{3t^2} f(1 + f)$;

but damping terms $\sim -\frac{4}{3t} \partial_t f \sim \frac{1}{t^2} f^{\frac{1}{2}}(1 + f)$.

Main Theorem shown by one picture



Rough expression of the main theorem

If the **initial inhomogeneities** $\|(\psi, \psi_0)\|_x$ are **sufficiently small**, then $t = t(t, x) \rightarrow t$, and $\varrho(t, x^k) \rightarrow f(t)$ and their derivatives as well of the reference ODEs, at least within a sufficiently large domain $\mathcal{D}$ close to $[t_0, t_m) \times \mathbb{R}^n$.

§4. Emergence of nonlinear Jean-type instabilities for QNLW
2. Ideas of the proofs

Ideas and Fuchsian direction

- Basic direction: the **Fuchsian** method (need **time compactifications** and **Fuchsian variables**)!

- Require **time compactifications** $[t_0, t_m) \rightarrow [-1, 0)$. How?

Idea: Intro. a compactified time like “Try 2”?

Difficulty: **Fail!** Since there is **no synchronized term** (**synchronizing source term synchronize the blowup time to 0**), and it is high possible that the solution **blows up at different time** (if it blows up!). The compactified time works only if the blow up time can be synchronized and the perturbations do not change the blowup times (if blowup at infinity, it may still work)

- Define the compactified time $\tau = g(t, x)$ for t by solving the equation

$$\partial_t g(t, x^i) = \frac{AB\varrho(t, x^i) (-g(t, x^i))^{\frac{2}{3A}+1}}{t^{\frac{2}{3}}(\varrho(t, x^i) + 1)^{\frac{1}{3}}},$$

$$g(t_0, x^i) = -1.$$

Compactified time $[t_0, t_m) \rightarrow [-1, 0)$

- **Overcome diff.:** Intro. two compactified time:
(1) for the reference solution (sol. to ref. ODE), use “try 2” compactified time;

$$\underbrace{\tau = \mathfrak{g}(t)}_{\text{Increasing}} = -\left(1 + \frac{2}{3}B \int_{t_0}^t s^{-\frac{2}{3}} f(s) (1 + f(s))^{-\frac{1}{3}} ds\right)^{-\frac{3A}{2}} \in [-1, 0).$$

It **synchronizes the blowup time** of the **reference solution**. However, the **perturbations may not blowup at this time**, blowup time may deviate it.

- In order to **be comparable** (this may **not hold!**), we intro the compactified time analogue to this

$$\tau = g(t, x^i) = -\left(1 + \frac{2}{3}B \int_{t_0}^t s^{-\frac{2}{3}} \varrho(s, x^i) (1 + \varrho(s, x^i))^{-\frac{1}{3}} ds\right)^{-\frac{3A}{2}} \in [-1, 0).$$

- **Wrong compactified time** leads wrong structures and **fails**. It is crucial how to choose it. **Need guess and experiments!**

Remark: Comparisons of ϱ and f

Wrong: $\varrho(t, x) - f(t)$ (when there is a synchronizing term as Try 2);

Correct: $\underline{\varrho}(\tau, \zeta^k) - \underline{f}(\tau)$ (align the variables by compactified time τ).

ODE equivalence of the compactification

The compactified time can be reexpressed in terms of two ODEs:

$$\partial_t g(t, x^i) = \frac{AB \varrho(t, x^i) (-g(t, x^i))^{\frac{2}{3A}+1}}{t^{\frac{2}{3}} (\varrho(t, x^i) + 1)^{\frac{1}{3}}},$$
$$g(t_0, x^i) = -1.$$

and

$$\partial_t g(t) = -A g(t) \frac{f(t)(f(t) + 1)}{t^2 f_0(t)} = \frac{AB f(t) (-g(t))^{1+\frac{2}{3A}}}{t^{\frac{2}{3}} (1 + f(t))^{\frac{1}{3}}},$$
$$g(t_0) = -1.$$

- These ODEs provide the Jacobian and determine how the coordinate transforms develop. We must **solve the variant of the main equation concurrently** with the **coordinate equation**.
- They provide some hidden identities.

The first coordinate transform

We express the main equation to a singular hyperbolic system (1st order) in terms of (τ, ζ) given by

$$\tau = g(t, x^i) \quad \text{and} \quad \zeta^i = x^i$$

Its **inverse transformation** denote

$$t = b(\tau, \zeta^i) \quad \text{and} \quad x^i = \zeta^i$$

and satisfies a ODE (**Why? Since it is Fuchsianable**)

$$\partial_\tau b(\tau, \zeta^i) = \frac{b^{\frac{2}{3}}(\tau, \zeta^i)(1 + \underline{\varrho}(\tau, \zeta^i))^{\frac{1}{3}}}{AB\underline{\varrho}(\tau, \zeta^i)(-\tau)^{\frac{2}{3A}+1}},$$

$$b(-1, \zeta^i) = t_0$$

- We do **not** give the coordinate transform directly but **give it by an evolution equation** (similar to the wave coordinates, perturbed Lagrangian coordinates, etc.)
- b and $\mathbf{b}_\zeta := \partial_\zeta \mathbf{b}$ become **unknown variables** since they describe the coordinate transform and this transform has been solved from an equation.

Singular symmetric hyperbolic system

- Intro. **perturbation variables**: e.g. $u(\tau, \zeta^k) = \frac{\underline{g}(\tau, \zeta^k) - \underline{f}(\tau)}{\underline{f}(\tau)}$
- Using a lot of **hidden relations** derived from the reference ODE and the quantities χ and ξ in “try 2” we can have a **singular symmetric hyperbolic equation** (similar to “try 2”).
- Comparing with “Try 2”, this method has already lead to the Fuchsian system and it is done! However, now it can **not be achieved**.

Lemma

$$\mathbf{A}^0 \partial_\tau U + \frac{1}{A_\tau} \mathbf{A}^i \partial_{\zeta^i} U = \frac{1}{A_\tau} \mathbf{A} U + \mathbf{F},$$

where $U := (u_0, u_j, u, \mathcal{B}_l, z)^T$, $\mathbf{F} = (\mathfrak{F}_{u_0}, \mathfrak{F}_{u_j}, \mathfrak{F}_u, \mathfrak{F}_{\mathcal{B}_l}, \mathfrak{F}_z)^T$,

$$\mathbf{A}^0 = \begin{pmatrix} 1 & \mathcal{R}^j & 0 & 0 & 0 \\ \mathcal{R}_k & (S + \mathcal{L})\delta_k^j & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \delta_s^l & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A}^i = \begin{pmatrix} 0 & H^{ij} & 0 & 0 & 0 \\ H_k^i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^i \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} -\frac{14}{3} + \mathcal{L}_{11} & -4kq^j + \mathcal{L}_{12}^j & 8 + \mathcal{L}_{13} & 0 & -8 + \mathcal{L}_{15} \\ 0 & (4k + \mathcal{L}_{22})\delta_k^j & 0 & (24k + \mathcal{L}_{24})\delta_k^l & 0 \\ -8 + \mathcal{L}_{31} & 0 & \frac{40}{3} + \mathcal{L}_{33} & 0 & -16 + \mathcal{L}_{35} \\ 0 & (\frac{2}{3} + \mathcal{L}_{42})\delta_s^j & 0 & (\frac{2}{3} + \mathcal{L}_{44})\delta_s^l & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Difficulty: $\mathbf{A}$ can **not be positive definite** whatever you do!

Ideas from algebraic observations

- $\tilde{\zeta}^1 = \frac{\alpha}{A} \ln(-\tau) + \zeta^1$ (where $\alpha > 0$) and $\tilde{\tau} = \tau$, we obtain the following transformations: $\partial_\tau = \partial_{\tilde{\tau}} + \frac{\alpha}{A\tilde{\tau}} \partial_{\tilde{\zeta}^1}$ and $\partial_{\zeta^1} = \partial_{\tilde{\zeta}^1}$.

$$\mathbf{A}^0 \partial_{\tilde{\tau}} U + \frac{1}{A\tilde{\tau}} \underbrace{(\alpha \mathbf{A}^0 + \mathbf{A}^1)}_{\substack{\alpha & 1 + \alpha U \\ 1 + \alpha U & \frac{1}{4}\alpha}} \partial_{\tilde{\zeta}^1} U = \frac{1}{A\tilde{\tau}} \mathbf{A} U + \mathbf{F}(U).$$

- A variable transformation $\mathfrak{U} = e^{\theta \tilde{\zeta}^1} U$ (where $\theta > 0$), the derivative transforms as follows: $\frac{1}{A\tilde{\tau}} \partial_{\tilde{\zeta}^1} U = e^{-\theta \tilde{\zeta}^1} \frac{1}{A\tilde{\tau}} \partial_{\tilde{\zeta}^1} \mathfrak{U} - \theta e^{-\theta \tilde{\zeta}^1} \frac{1}{A\tilde{\tau}} \mathfrak{U}$.

$$\begin{aligned} & \underbrace{\begin{pmatrix} 1 & e^{-\theta \tilde{\zeta}^1} \mathfrak{U} \\ e^{-\theta \tilde{\zeta}^1} \mathfrak{U} & \frac{1}{4} \end{pmatrix}}_{=:\mathbf{B}^0} \partial_{\tilde{\tau}} \mathfrak{U} + \frac{1}{A\tilde{\tau}} \underbrace{\begin{pmatrix} \alpha & 1 + \alpha e^{-\theta \tilde{\zeta}^1} \mathfrak{U} \\ 1 + \alpha e^{-\theta \tilde{\zeta}^1} \mathfrak{U} & \frac{1}{4}\alpha \end{pmatrix}}_{=:\mathbf{B}^1} \partial_{\tilde{\zeta}^1} \mathfrak{U} \\ &= \frac{1}{A\tilde{\tau}} \underbrace{\begin{pmatrix} \alpha\theta - \frac{14}{3} & \theta + \alpha\theta e^{-\theta \tilde{\zeta}^1} \mathfrak{U} - q \\ \theta + \alpha\theta e^{-\theta \tilde{\zeta}^1} \mathfrak{U} & 1 + \frac{1}{4}\alpha\theta \end{pmatrix}}_{=:\mathbf{B}} \mathfrak{U} + e^{\theta \tilde{\zeta}^1} \mathbf{F}(e^{-\theta \tilde{\zeta}^1} \mathfrak{U}). \end{aligned}$$

Remark

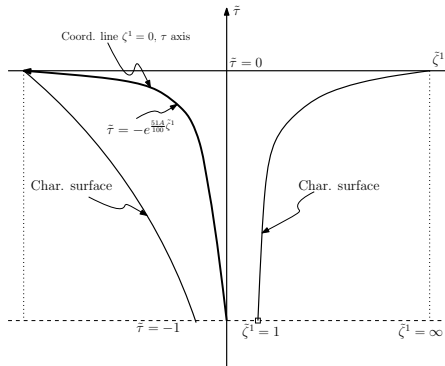
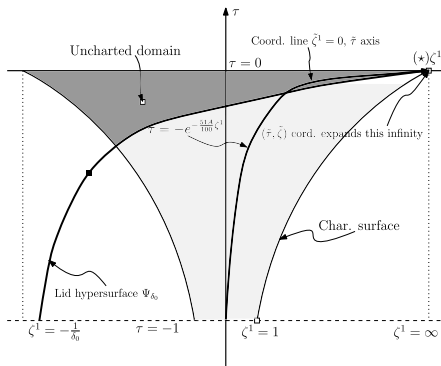
An intuitive idea to overcoming this difficulty is to solve a global existence problem for a revised hyperbolic system and ensure that this revised system is consistent with the main eq. (or its variant) within a sufficiently large lens-shaped domain. Outside the lens-shaped domain, we have considerable flexibility to modify the hyperbolic equation to align with the Fuchsian formulations, allowing us to use the Fuchsian GIVP for the revised system to achieve a global solution.

Two revisions (geometric meanings)

(1) The second coordinate transform: the tilted coordinate

$$\tilde{\tau} = \tilde{\tau}(\tau, \zeta^k) = \tau \quad \text{and} \quad \tilde{\zeta}^i = \tilde{\zeta}^i(\tau, \zeta^k) = \frac{ac^i}{A} \ln(-\tau) + \zeta^i,$$

- **Motivations:** Expand the “null infinity” (not precisely) and upright a timelike direction (close to null) to be the time axis. since our analysis can only work in this “closed to null” domain.
- From the equation point of view, (1) generate more terms in $\frac{1}{A\tau} \mathbf{A}^i$ and will help compensate $\frac{1}{A\tau} \mathbf{A}$ to achieve the positive definiteness.
- From the geometric point of view, they tilt the characteristic conoid and expand the “near-null” domain.



(2) rescale all the variables by **spatial factors**, e.g., $\mu(\tilde{\zeta}^1) := \sigma_0 e^{-\frac{153}{\delta_0}} e^{-51\tilde{\zeta}^1}$ and the variable, e.g., becomes

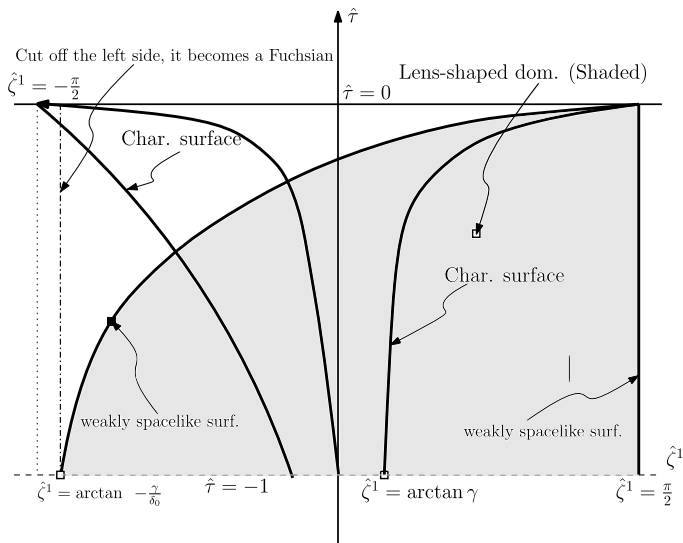
$$u_0(\tilde{\tau}, \tilde{\zeta}) = \frac{1}{\sigma_0 e^{-\frac{153}{\delta_0}} e^{-51\tilde{\zeta}^1}} \tilde{u}_0(\tilde{\tau}, \tilde{\zeta})$$

- **Motivation:** **Spatial factors** like μ will **separate a new singular remainder term** $\frac{1}{A\tau} \mathbf{A}_{\text{remainder}} U$ from $\frac{1}{A\tau} \mathbf{A}^i \partial_i U$, and $\frac{1}{A\tau} \mathbf{A}_{\text{remainder}} U$ **compensate** $\frac{1}{A\tau} \mathbf{A} U$ to obtain a **positive definite singular** lower order term (consists with the Fuchsian).
- Along the boundary of the char. cone, $\tilde{\tau} \sim e^{-\tilde{\zeta}^1}$ gives **decay factors**.
- **Defect:** $\mu \sim e^{-51\tilde{\zeta}^1}$ introduce infinities to the equation as $\tilde{\zeta}^1 \rightarrow -\infty$. Break the structures.
- **Idea to overcome:** **Revise** the equation **by cutoff function** ϕ such that the **infinities vanish**. However, the equation **fails to equivalent** to the original equation due to the revision.

$$\phi \in C^\infty(\mathbb{R}; [0, 1]), \quad \phi|_{[-\delta_0^{-1}, +\infty)} = 1 \quad \text{and} \quad \text{supp} \phi \subset [-2\delta_0^{-1}, +\infty) \subset \mathbb{R}.$$

How to recover the solution of the original one?

- To recover original solution, only use the **lens-shaped domain** (determination domain, see Fig. to explain)

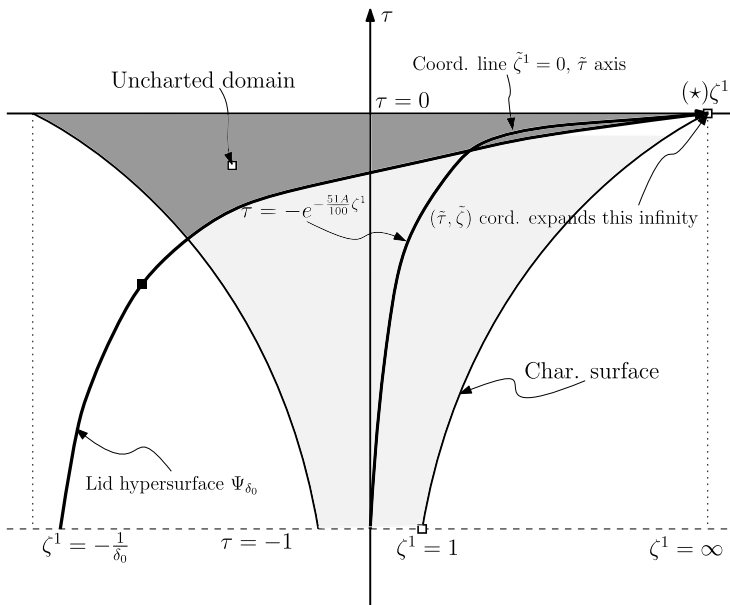


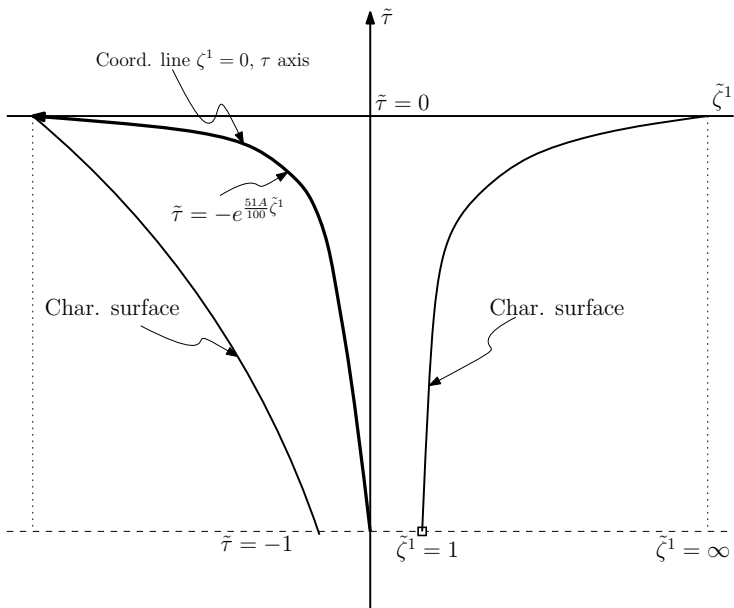
The revised system becomes a Fuchsian system by compactifying space

The third coordinate transform (**compactifying the space**)

$$\hat{\tau} = \tilde{\tau} \in [-1, 0) \quad \text{and} \quad \hat{\zeta}^i = \arctan(\gamma \tilde{\zeta}^i) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- $\mathbb{R}^n$ becomes $\mathbb{T}^n$, a **closed manifold** which is required by the Fuchsian analysis.
- After this coordinate transform, we have **Fuchsian formulation** and can derive the global existence and stability result for this revised system.
- Using determination domain obtain the main theorem.





*Thank you
for your attention!*