FLRW stability for polytropic (Makino) fluids
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Einstein-Euler system with $\Lambda > 0$

- Einstein-Euler equation with a positive cosmological constant $\Lambda > 0$ is given by

$$\tilde{G}^{\mu\nu} + \Lambda \tilde{g}^{\mu\nu} = \tilde{T}^{\mu\nu},$$
$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0,$$

  where $\tilde{G}^{\mu\nu} = \tilde{R}^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ is the Einstein tensor of the metric $\tilde{g} = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$, and $\tilde{T}^{\mu\nu} = (\rho + p)\tilde{u}^\mu \tilde{u}^\nu + p\tilde{g}^{\mu\nu}$ is the perfect fluid stress-energy tensor (Assume $\tilde{u}^\mu \tilde{u}_\mu = -1$).
- 14 independent equations. 10 metric variables and 5 fluid variables ρ, p and $\tilde{u}^i$. Need to <u>assume</u> a *isentropic* equation of state (EOS) $p = f(\rho)$ to complete the system.
- equation of state

$$p = f(\rho) \begin{cases} := K\rho, & 0 < K \leq \frac{1}{3}. \\ := \underline{K\rho^{\frac{n+1}{n}}}, -\frac{\Lambda^{1+\vartheta}}{(\rho+\Lambda)^\vartheta} + \Lambda, \ldots \end{cases}$$

FLRW metric

- Solving Friedmann equations (Einstein eqs under assumptions the universe is homogeneous, isotropic, expanding and perfect fluids-filled) yields FLRW metric on $\mathcal{M} = (0, 1] \times \mathbb{T}^3$,

$$\begin{aligned}
\widetilde{\eta} =& \frac{1}{\tau^2}\Big(-\frac{1}{\omega^2(\tau)} d\tau^2 + \delta_{ij} dx^i dx^j\Big),\\
\tilde{u} =& -\tau\omega\partial_\tau\\
\tau^4\bar{\rho}(1) \leq& \bar{\rho}(\tau) \leq \tau^3\bar{\rho}(1)\\
\frac{1}{3}\bar{\rho}(1)\tau^4 \leq& \omega^2 - \frac{\Lambda}{3} \leq \frac{1}{3}\bar{\rho}(1)\tau^3.
\end{aligned}$$

- The standard FLRW metric is

$$\eta = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j,$$

We have time transform $a(t) = \frac{1}{\tau}$ $(\tau \in (0, 1])$ and $\frac{dt}{d\tau} = -\frac{1}{\tau\omega(\tau)}$, then

$$t = -\int_1^\tau \frac{1}{y\omega(y)} dy > 0.$$

Motivations and Questions

- Stability problem: Small inputs $\Rightarrow$ small outputs. Usually, small perturbations of data $\Rightarrow$ small perturbations of solutions.
- We have extra inputs "equation of state (EOS)", so we want to know how EOS affects solutions, i.e., vary the equation of state $f(\rho)$ slightly, how the solutions react?
- Try to find a function space

$$\mathcal{H}_p := \{f : \text{some properties}\}$$

  with some topology, such that for any $\epsilon > 0$, there is $\delta > 0$, such that the initial data $\|u_0 - \tilde{u}_0\| < \delta$ and $\|f - \tilde{f}\|_{\mathcal{H}} < \delta$, then $\|u - \tilde{u}\| < \epsilon_0$.
- $\Lambda > 0$ and $p = K\rho$ leads to exponential decay, then stability. Directly remove $\Lambda > 0$ may be difficult to see what happens. We want to gradually use "worse" EOS to offset some effect of de Sitter-like expansion and see how the solutions react. by knowing this, we may find some clues on the conditions of blowup.

- Try to construct a large class of fluids, Makino fluids, which the de-Sitter like behaviors holds. Makino fluids include many well-known fluids like polytropic fluids, Chaplygin gas, linear EOS fluids, etc.
- This talk focus on the polytropic fluids which have worse behavior than the linear one (in fact, from the point view of the proof, the Chaplygin gas is exactly the same as the linear one).

Brief history

- Brauer, Rendall and Reula (1994), nonlinear stability of Newtonian cosmological model. (polytropic)
- Rinström (2008), NL stability of Einstein-scalar field.
- Rodnianski and Speck (2013,2014), NL stability of FLRW with $\Lambda > 0$ and linear EOS for $0 < K < 1/3$.
- Lübble and Kroon (2013), NL stability of FLRW with $\Lambda > 0$ and linear EOS for $K = 1/3$.
- Hadžić and Speck (2015), ...for $K = 0$.
- Oliynyk (2016), NL stability of FLRW with $\Lambda > 0$ and linear EOS for $0 < K \leq 1/3$ by using Fuchsian type system.
- L. and Oliynyk (2017,2018), Cosmological Newtonian limits on large scales.
- Lefloch and Wei (2016), NL stability of FLRW with $\Lambda > 0$ and Chaplygin gases by using Fuchsian type system

Main theorem

Main Theorem

Suppose $k \in \mathbb{Z}_{\geq 3}$, $\varepsilon \in (0, 1]$, $\Lambda > 0$, $g_0^{\mu\nu} \in H^{k+1}(\mathbb{T}^3)$, $g_1^{\mu\nu}$, ρ_0, $\nu^\alpha \in H^k(\mathbb{T}^3)$, $\rho_0 > 0$ for all $x \in \mathbb{T}^3$ and the unknowns are determined by the data on the initial hypersurface that $(g^{\mu\nu}, \partial_\tau g^{\mu\nu}, \rho, u^i)|_{\tau=1} = (g_0^{\mu\nu}, g_1^{\mu\nu}, \rho_0, \nu^i)$ which solves the constraint equations

$$(G^{0\mu} - T^{0\mu})|_{\tau=1} = 0 \quad \text{and} \quad Z^\mu|_{\tau=1} = 0,$$

and require $n \in (\frac{3-\varepsilon}{2}, 3-\varepsilon] \cap (1, 3)$ and

$$0 < \bar{\rho}(1) \leq \frac{1}{(4Kn(n+1))^n}\left(2n\sqrt{\frac{1}{3}\left(1-\frac{3-\epsilon}{2n}\right)}\right)^{2n} \quad \text{(debts 1)}$$

Then there exists a constant $\sigma > 0$, such that if

$$\|g_0^{\mu\nu} - \eta^{\mu\nu}(1)\|_{H^{k+1}} + \|g_1^{\mu\nu} - \partial_\tau \eta^{\mu\nu}(1)\|_{H^k} + \|\rho_0 - \bar{\rho}(1)\|_{H^k} + \|\nu^i\|_{H^k} < \sigma,$$

there exists a unique classical solution $g^{\mu\nu} \in C^2((0, 1] \times \mathbb{T}^3)$, ρ, $v^i \in C^1((0, 1] \times \mathbb{T}^3)$ to the conformal Einstein-Euler system that satisfies the initial data, the wave gauge $Z^\mu = 0$ in $(0, 1] \times \mathbb{T}^3$ and the following regularity conditions

$$(g^{\mu\nu}, u^\mu, \rho) \in \bigcap_{\ell=0}^{2} C^\ell((0, 1], H^{k+1-\ell}(\mathbb{T}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((0, 1], H^{k-\ell}(\mathbb{T}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((0, 1], H^{k-\ell}(\mathbb{T}^3)),$$

and the estimates that

$$\|g^{\mu\nu}(\tau) - \eta^{\mu\nu}(\tau)\|_{H^{k+1}} + \|\partial_\kappa g^{\mu\nu}(\tau) - \partial_\kappa \eta^{\mu\nu}(\tau)\|_{H^k} + \|\rho(\tau) - \bar{\rho}(\tau)\|_{H^k} + \|u^i(\tau)\|_{H^k} \lesssim \sigma.$$

Discussion

- Does that density requirement related to Jeans' mass for star formations? $M_J \sim \bar{\rho}^{-1/2}$? It does not appear in linear EOS.
- If, roughly speaking, EOS can be expand to

$$p(\rho) = K\rho + p_2(\rho)$$

around $\rho = 0$. Then always can prove stability for small perturbations. so it is stable for small perturbation of EOS in this set with $\|\cdot\|_{L^\infty}$. If EOS can be expand to

$$p = K\rho^{1+\frac{1}{n}} + p_2(\rho)$$

when $n \to \infty$ (we need $n \in (1,3)$), we do not have answer. If above function set include this type of EOS, we do not know the stability w.r.t small EOS perturbations.

Tool: Fuchsian formulations (Singular hyperbolic system) of Einstein–Euler system

$$B^\mu \partial_\mu u = \frac{1}{t}\mathbf{BP}u + H \quad \text{in } [-1,0)\times\mathbb{T}^n,$$
$$u = u_0 \quad \text{on } \{-1\}\times\mathbb{T}^n.$$

Main assumptions of this system

1. $\mathbf{P}$ is a constant, symmetric projection operator (Pick up decay variables by $\mathbf{P}$).
2. $\frac{1}{\gamma_1}\mathbb{I} \leq \mathring{B}^0 \leq \frac{1}{\kappa}\mathring{\mathbf{B}} \leq \gamma_2\mathbb{I}$ (Give right signs and determine dissipative effects).
3. $(B^\mu)^T = B^\mu, \quad [\mathbf{P},\mathbf{B}] = \mathbf{PB} - \mathbf{BP} = 0.$
4. $\mathbf{P}^\perp B^0(t,\mathbf{P}^\perp u)\mathbf{P} = \mathbf{P}B^0(t,\mathbf{P}^\perp u)\mathbf{P}^\perp = 0.$
5. $|\mathbf{P}^\perp[D_u B^0(t,u)(B^0)^{-1}\mathbf{BP}u]\mathbf{P}^\perp|_{op} \leq \alpha|t| + \beta|\mathbf{P}u|^2.$

$(3,4,5$ gives how the variables coupling to each other$)$, and some regularity assumptions on the coefficients and remainders. Advantage: allow suitable coupling of variables.

Theorem of singular system

Theorem (Oliynyk, 2016)

Suppose that $k \geq \frac{n}{2} + 1$, $u_0 \in H^k(\mathbb{T}^n)$ and above conditions are fulfilled. Then there exists a $T_* \in (-1, 0)$, and a unique classical solution $u \in C^1([-1, T_*] \times \mathbb{T}^n)$ that satisfies $u \in C^0([-1, T_*], H^k) \cap C^1([-1, T_*], H^{k-1})$ and the energy estimate

$$\|u(t)\|_{H^k}^2 - \int_{-1}^{t} \frac{1}{\tau} \|\mathbf{P}u\|_{H^k}^2 d\tau \leq Ce^{C(t+1)}(\|u(-1)\|_{H^k}^2)$$

for all $-1 \leq t < T_*$, where $C = C(\|u\|_{L^\infty([-1,T_*),H^k)}, \gamma_1, \gamma_2, \kappa)$, and can be uniquely continued to a larger time interval $[T_0, T^*)$ for all $T^* \in (T_*, 0]$ provided $\|u\|_{L^\infty([-1,T_*),W^{1,\infty})} < \infty$.

Reduced conformal Einstein equations (Review)

Conformal transform

$$g_{\mu\nu} = e^{-2\Phi}\tilde{g}_{\mu\nu} \quad \text{and} \quad u^\mu = e^{\Phi}\tilde{u}^\mu$$

The conformal Einstein equation becomes

$$\begin{aligned} -2R^{\mu\nu} + 2\nabla^{(\mu}Z^{\nu)} + A^{\mu\nu}_{\kappa}Z^\kappa = & -4\nabla^\mu\nabla^\nu\Phi + 4\nabla^\mu\Phi\nabla^\nu\Phi \\ & -2\left[\Box_g\Phi + 2|\nabla\Phi|^2_g + \left(\frac{\rho - p}{2} + \Lambda\right)e^{2\Phi}\right]g^{\mu\nu} - 2e^{2\Phi}(\rho + p)u^\mu u^\nu, \end{aligned}$$

where $X^\mu := \Gamma^\mu - \gamma^\mu$,

$$A^{\mu\nu}_\kappa = -X^{(\mu}\delta^{\nu)}_\kappa + Y^{(\mu}\delta^{\nu)}_\kappa \quad \text{and} \quad Z^\mu = X^\mu + Y^\mu$$

3 key choices of this transform

- Conformal factor: $\Phi = -\ln(\tau)$ (partially include decay informations);
- Wave gauge: $Z^\mu = 0$ and the source function $Y^\mu := -2(g^{\mu\nu} - \eta^{\mu\nu})\nabla_\nu\Phi$; (Kill redundant high order singular terms by costing nothing)
- Variables:

Gravitational variables

$$
\begin{aligned}
\mathbf{q} =& g^{00} - \eta^{00} + \frac{\eta^{00}}{3} \ln(\det(g^{ij})), \\
\mathbf{u}^{0\nu} =& \frac{g^{0\nu} - \eta^{0\nu}}{2\tau}, \\
\mathbf{u}^{0\nu}_0 =& \partial_\tau (g^{0\nu} - \eta^{0\nu}) - \frac{3(g^{0\nu} - \eta^{0\nu})}{2\tau}, \\
\mathbf{u}^{0\nu}_i =& \partial_i (g^{0\nu} - \eta^{0\nu}), \\
\mathbf{u}^{ij} =& \mathbf{g}^{ij} - \delta^{ij}, \\
\mathbf{u}^{ij}_\mu =& \partial_\mu \mathbf{g}^{ij}, \\
\mathbf{u} =& \mathbf{q}, \\
\mathbf{u}_\mu =& \partial_\mu \mathbf{q}.
\end{aligned}
$$

where $\mathbf{g}^{ij} = \det(\check{g}_{lm})^{\frac{1}{3}} g^{ij}$ and $\check{g}_{lm} = (g^{lm})^{-1}$.

- Suitable combinations kill higher order singular terms
- $1/\tau$ coeff. package more decays in the variables
- The constant coeff. are also crucial for killing bad terms

The Einstein equations (in the form of 1st order equation) become

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0^{0\mu} \\ \mathbf{u}_j^{0\mu} \\ \mathbf{u}^{0\mu} \end{pmatrix} = \frac{1}{\tau} \mathbf{A} \mathbf{P}^\star \begin{pmatrix} \mathbf{u}_0^{0\mu} \\ \mathbf{u}_j^{0\mu} \\ \mathbf{u}^{0\mu} \end{pmatrix} + F_1,$$

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0^{lm} \\ \mathbf{u}_j^{lm} \\ \mathbf{u}^{lm} \end{pmatrix} = \frac{1}{\tau} (-2g^{00}) \Pi \begin{pmatrix} \mathbf{u}_0^{lm} \\ \mathbf{u}_j^{lm} \\ \mathbf{u}^{lm} \end{pmatrix} + F_2,$$

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_j \\ \mathbf{u} \end{pmatrix} = \frac{1}{\tau} (-2g^{00}) \Pi \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_j \\ \mathbf{u} \end{pmatrix} + F_3,$$

where

$$A^0 = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & g^{ij} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix}, \quad A^k = \begin{pmatrix} -2g^{0k} & -g^{jk} & 0 \\ -g^{ik} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & \frac{3}{2} g^{li} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix},$$

$$\mathbf{P}^\star = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \delta_l^j & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 6\mathbf{u}^{0i}\mathbf{u}_i^{0\mu} + 4\mathbf{u}^{00}\mathbf{u}_0^{0\mu} - 4\mathbf{u}^{00}\mathbf{u}^{0\mu} + \mathfrak{H}^{0\mu} \\ 0 \\ 0 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 4\mathbf{u}^{00}\mathbf{u}_0^{ij} + \mathfrak{R}^{ij} \\ 0 \\ -g^{00}\mathbf{u}_0^{lm} \end{pmatrix}, \quad F_3 = \begin{pmatrix} 4\mathbf{u}^{00}\mathbf{u}_0 - 8(\mathbf{u}^{00})^2 + \mathfrak{F} \\ 0 \\ -g^{00}\mathbf{u}_0 \end{pmatrix}.$$

The dangerous terms in remainders F_ℓ: $\frac{1}{\tau^2}(\rho - \bar{\rho})$.

Reduced conformal Euler equations (New)

Aim

Transform conformal Euler equations

$$\nabla_\mu \widetilde{T}^{\mu\nu} = -6\widetilde{T}^{\mu\nu}\nabla_\mu\Phi + g_{\kappa\lambda}\widetilde{T}^{\kappa\lambda}g^{\mu\nu}\nabla_\mu\Phi.$$

to the singular hyperbolic system. Four Steps.

Step 1: symmetric hyperbolic system (Standard)

$3+1$ decomposition of above equation. Define an operator

$$L^\mu_i = \delta^\mu_i - \frac{u_i}{u_0}\delta^\mu_0 \quad \text{and} \quad M_{ki} = g_{ki} - \frac{u_i}{u_0}g_{0k} - \frac{u_k}{u_0}g_{0i} + \frac{u_i u_k}{u_0^2}g_{00}.$$

Then Euler equation, using $u_\mu \nabla_\nu u^\mu = 0$ (that is $\nabla_\nu u^0 = -\frac{u_i}{u_0}\nabla_\nu u^i$), can be rewrite to ($s^2 = dp/d\rho \sim K(1+\frac{1}{n})\rho^{\frac{1}{n}}$)

$$\begin{aligned} u^\mu\partial_\mu\rho + (\rho+p)L^\mu_i\nabla_\mu u^i &= -3(\rho+p)u^\mu\nabla_\mu\Phi, \\ \frac{s^2 L^\mu_i}{\rho+p}\partial_\mu\rho + M_{ki}u^\mu\nabla_\mu u^i &= -L^\mu_k\nabla_\mu\Phi \end{aligned}$$

Step 2: non-degenerated symmetric hyperbolic system

Problem: can not be symmetrized by simply multiplying some factor, for example, $(a) \times \frac{s^2}{\rho+p}$ and $(b) \times (\rho+p)$, or $(a) \times s^2$ and $(b) \times (\rho+p)^2$, etc.

$$
\begin{aligned}
(a) \quad & u^\mu \partial_\mu \rho + (\rho+p) L^\mu_i \nabla_\mu u^i = -3(\rho+p) u^\mu \nabla_\mu \Phi, \\
(b) \quad & \frac{s^2 L^\mu_i}{\rho+p} \partial_\mu \rho + M_{ki} u^\mu \nabla_\mu u^i = -L^\mu_k \nabla_\mu \Phi
\end{aligned}
$$

where $s^2 = dp/d\rho \sim K(1+\frac{1}{n})\rho^{\frac{1}{n}} \sim \tau^{3/n} \searrow 0$, $n \in (1,3)$. The coefficient matrix blows up (unbounded) or goes to 0 (degenerated) as $\tau \to 0$. Hyperbolicity is destroyed

Solving this problem

Makino's idea: Try to find a transform $\rho \mapsto \alpha$ and choose λ to symmetrize the system.

$$(a) \quad \lambda^2 \frac{d\alpha}{d\rho} u^\mu \partial_\mu \rho + \lambda^2 \frac{d\alpha}{d\rho} (\rho + p) L_i^\mu \nabla_\mu u^i = -3(\rho + p) \lambda^2 \frac{d\alpha}{d\rho} u^\mu \nabla_\mu \Phi,$$

$$(b) \quad \frac{s^2 L_i^\mu}{\rho + p} \partial_\mu \rho + M_{ki} u^\mu \nabla_\mu u^i = - L_k^\mu \nabla_\mu \Phi$$

Multiplying $\lambda^2 \frac{d\alpha}{d\rho}$ on (a) and take λ such that

$$\lambda^2 (\rho + p) \frac{d\alpha}{d\rho} = \frac{s^2}{\rho + p} \frac{d\rho}{d\alpha}.$$

Polytropic fluids $p = K\rho^{1+\frac{1}{n}}$, we usually choose Makino's density

$$\rho = \frac{1}{\big(4Kn(n+1)\big)^n} \alpha^{2n} \quad \text{and} \quad \lambda = \Big(1 + \frac{1}{4n(n+1)} \alpha^2\Big)^{-1}.$$

Then the Euler equations become a non-degenerated form (after subtracting the background solution)

$$\lambda^2 u^\mu \partial_\mu(\alpha - \bar{\alpha}) + \lambda s L^\mu_i \partial_\mu u^i = \frac{3}{\tau}\lambda s u^0 - \frac{3}{\tau}\lambda^2 u^0 \bar{q} - \lambda s L^\mu_i \Gamma^i_{\mu\nu} u^\nu,$$
$$\lambda s L^\mu_i \partial_\mu(\alpha - \bar{\alpha}) + M_{ki} u^\mu \partial_\mu u^k = L^0_i \frac{1}{\tau} - \frac{3}{\tau}\lambda s L^0_i \bar{q} - M_{ki} u^\mu \Gamma^k_{\mu\nu} u^\nu.$$

where $\bar{q} := q(\bar{\alpha}) = \bar{s}/\bar{\lambda}$.

Direct Problem:

This equation is problematic when considering the Einstein-Euler system together to get the singular model eqs, because, recalling the dangerous term $\frac{1}{\tau^2}(\rho - \bar{\rho})$ in F_ℓ of Einsten eqs. and $\rho \sim \alpha^{2n}$, the variable $\alpha - \bar{\alpha}$ can lead a $1/\tau^2$ singular term in Einstein eqs.

Solving problem:

- Rescale α by a new weighted Makino variable $\zeta := \beta(\tau)^{-1}\alpha$;
- In order to keep the symmetry of coef. matrix, we have to define a new weighted velocity $v^i := \beta(\tau)^{-1} u^i$.

Using weighted variables ζ and v^i, Euler eqs become

$$\begin{aligned} \lambda^2 u^\mu \partial_\mu \delta\zeta + \lambda s L_i^\mu \partial_\mu v^i =& S, \\ \lambda s L_i^\mu \partial_\mu \delta\zeta + M_{ki} u^\mu \partial_\mu v^k =& S_i, \end{aligned}$$

What $\beta(\tau)$ is? (debts 2)

We can calculate the background

$$\bar{\alpha}(\tau) = \tau^{\frac{3}{2n}} \left(\frac{1}{\bar{\alpha}^2(1)} + \frac{1}{4n(n+1)} - \frac{1}{4n(n+1)} \tau^{\frac{3}{n}} \right)^{-\frac{1}{2}} \sim \tau^{\frac{3}{2n}}$$

and $\rho \sim \alpha^{2n}$, therefore, we can take $\beta \sim (\tau^{\frac{2}{2n}}, \tau^{\frac{3}{2n}})$ to make sure the dangerous term $\frac{1}{\tau^2}(\rho - \bar{\rho})$ in F_ℓ of Einsten eqs. is regular in τ and also make sure ζ is bounded or with a small decay.

Step 3: non-degenerated symmetric hyperbolic formulations of weighted variables

Using weighted variables ζ and v^i, Euler eqs become

$$\begin{aligned}\lambda^2 u^\mu \partial_\mu \delta\zeta + \lambda s L^\mu_i \partial_\mu v^i =& S,\\ \lambda s L^\mu_i \partial_\mu \delta\zeta + M_{ki} u^\mu \partial_\mu v^k =& S_i,\end{aligned}$$

where

$$\begin{aligned}S =& \frac{1}{\tau}\lambda^2 u^0 \left[3q'(\bar{\alpha}) + \frac{3}{2}q''(\alpha_{K_7})\beta\delta\zeta - \chi(\tau)\right]\delta\zeta + \frac{1}{\tau}\chi(\tau)\left(\frac{\lambda s g_{ij}\beta(\tau)v^j}{u_0}\right)v^i \\ & + \frac{\lambda s \beta'(\tau) g_{0i} u^0}{\beta(\tau) u_0} v^i - \lambda \frac{s}{\beta(\tau)}\frac{u^0}{2} g^{ik}(\partial_i g_{k0} + \partial_\tau g_{ki} - \partial_k g_{i0}) - \mathtt{S}(\tau, \mathbf{U}) \text{ regular} \\ S_i =& \frac{1}{\tau}\left(-\frac{g_{ik}}{u_0}\left(1 - 3s^2\frac{\bar{q}}{q} - \chi(\tau)\beta(\tau)\lambda s \delta\zeta\right) - \chi(\tau) M_{ki} u^0\right) v^k \\ & -2\left(\frac{3(\lambda s - \bar{\lambda}\bar{s})\bar{s}}{\bar{\lambda}\beta(\tau)} + \chi(\tau)\lambda s \delta\zeta\right) g_{ij}\mathbf{u}^{0j} - \frac{1}{\tau}\frac{\tau g_{ij}}{\beta(\tau)}\left((1+6\bar{s}^2)\mathbf{u}^{0j} + \mathbf{u}^{0j}_0\right) \\ & + \frac{\eta_{00}\mathbf{u}^{00}_i}{2\beta(\tau)} + \mathtt{S}_i(\tau, \mathbf{U}, \mathbf{V})\end{aligned}$$

and $\chi(\tau) := \tau\partial_\tau \ln\beta(\tau)$.

Problem:

$$S_i = \frac{1}{\tau}\left(-\frac{g_{ik}}{u_0}\left(1 - 3s^2\frac{\bar{q}}{q} - \chi(\tau)\beta(\tau)\lambda s\delta\zeta\right) - \chi(\tau)M_{ki}u^0\right)v^k$$
$$-\frac{1}{\tau}\frac{\tau g_{ij}}{\beta(\tau)}\left((1+6\bar{s}^2)\mathbf{u}^{0j} + \mathbf{u}_0^{0j}\right) + \frac{\eta_{00}\mathbf{u}_i^{00}}{2\beta(\tau)} + \hat{\mathrm{S}}_i(\tau, \mathbf{U}, \mathbf{V})$$

- involves $\mathbf{u}^{0j}$, $\mathbf{u}_0^{0j}$, $\mathbf{u}_i^{00}$, but they are coupled in a bad way (I will show you how bad after I give you the right form).
- Now the purpose is to adjust the coefficients in front of $\mathbf{u}^{0j}$ and $\mathbf{u}_0^{0j}$ to get a good coupling way: the factor $1\times\mathbf{u}^{0j} + 1\times\mathbf{u}_0^{0j}$ (GOOD!) rather than $(1+6\bar{s}^2)\times\mathbf{u}^{0j} + 1\times\mathbf{u}_0^{0j}$ (BAD!). Why? Later (debt 3).

Step 4: non-degenerated symmetric hyperbolic formulations of new "better" variables

Solving problem: We introduce a new variable

$$\mathbf{v}^k = v^k - Ag^{0k} = v^k - 2\tau A\mathbf{u}^{0k},$$

where

$$A = A(\tau) = -\frac{3\bar{s}^2}{\omega\beta(\tau)}.$$

Euler eqs become

$$\begin{aligned}
\lambda^2 u^\mu \partial_\mu \delta\zeta + \lambda s L^\mu_i \partial_\mu \mathbf{v}^i =& \frac{1}{\tau}\lambda^2 u^0 \left[3q'(\bar{\alpha}) + \frac{3}{2}q''(\alpha_{K_7})\beta\delta\zeta - \chi(\tau)\right]\delta\zeta \\
&+ \frac{\chi(\tau)}{\tau}\left(\frac{\beta(\tau)\lambda s g_{ij}\mathbf{v}^j}{u_0}\right)\mathbf{v}^i + \hat{\mathrm{F}}(\tau, \mathbf{U}, \tilde{\mathbf{V}}) \\
\lambda s L^\mu_i \partial_\mu \delta\zeta + M_{ki} u^\mu \partial_\mu(\mathbf{v}^k) =& \frac{1}{\tau}\left(-\frac{g_{ik}}{u_0}\left(1 - 3s^2\frac{\bar{q}}{q} - \chi(\tau)\beta(\tau)\lambda s\delta\zeta\right) - \chi(\tau)g_{ki}u^0\right)\mathbf{v}^k \\
&- \frac{g_{ij}}{\tau}\left(\frac{\tau}{\beta(\tau)}(1 - 3\bar{s}^2)(\mathbf{u}^{0j}_0 + \mathbf{u}^{0j})\right) + \frac{1}{\tau}\frac{\tau\eta_{00}\mathbf{u}^{00}_i}{2\beta(\tau)} \\
&+ \hat{\mathrm{S}}_i(\tau, \mathbf{U}, \tilde{\mathbf{V}})
\end{aligned}$$

Final reduced Euler equation

$$N^\mu \partial_\mu \tilde{\mathbf{V}} = \frac{1}{\tau}\mathbf{N}\mathbf{P}^\dagger \tilde{\mathbf{V}} + \frac{1}{\tau}(\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu)\mathbf{U}^\mu + F(\tau, \tilde{\mathbf{V}}, \mathbf{U}),$$

where

$$\tilde{\mathbf{V}} = (\delta\zeta, \mathbf{v}^p)^T, \qquad \mathbf{U}^\mu = (\mathbf{u}^{0\mu}_0, \mathbf{u}^{0\mu}_j, \mathbf{u}^{0\mu})^T$$

and

$$N^\mu = \begin{pmatrix} \lambda^2 u^\mu & \lambda s L^\mu_p \\ \lambda s L^\mu_r & M_{rp} u^\mu \end{pmatrix}, \quad \mathbf{E}_0 = \frac{\tau\eta_{00}}{2\beta(\tau)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^j_r & 0 \end{pmatrix},$$

$$\mathbf{N} = \begin{pmatrix} \lambda^2 u^0 \left[3q'(\bar{\alpha}) + \frac{3}{2}q''(\alpha_{K_7})\beta\delta\zeta - \chi(\tau)\right] & \frac{1}{u_0}\chi(\tau)\beta(\tau)\lambda s g_{ij}\mathbf{v}^j \\ \frac{1}{u_0}\chi(\tau)\beta(\tau)\lambda s g_{rj}\mathbf{v}^j & -\frac{g_{ir}}{u_0}\left(1 - 3s^2\frac{\bar{q}}{q}\right) - \chi(\tau)g_{ri}u^0, \end{pmatrix},$$

$$\mathbf{E}_q = -\frac{\tau}{\beta(\tau)}(1 - 3\bar{s}^2)\begin{pmatrix} 0 & 0 & 0 \\ g_{rq} & 0 & g_{rq} \end{pmatrix}, \quad \mathbf{P}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & \delta^i_p \end{pmatrix}.$$

and $F = (\hat{\mathrm{F}}, \hat{\mathrm{S}}_i)^T$.

Complete reduced Einstein–Euler system

Let

$$\mathbf{U} := (\mathbf{u}_0^{0\mu}, \mathbf{u}_j^{0\mu}, \mathbf{u}^{0\mu}, \mathbf{u}_0^{lm}, \mathbf{u}_j^{lm}, \mathbf{u}^{lm}, \mathbf{u}_0, \mathbf{u}_j, \mathbf{u})^T \quad \text{and} \quad \tilde{\mathbf{V}} = (\delta\zeta, \mathbf{v}^p)^T,$$

to get the complete non-degenerated singular symmetric hyperbolic system

$$B^\mu \partial_\mu \begin{pmatrix} \mathbf{U} \\ \tilde{\mathbf{V}} \end{pmatrix} = \frac{1}{\tau} \mathbf{B}\mathbf{P} \begin{pmatrix} \mathbf{U} \\ \tilde{\mathbf{V}} \end{pmatrix} + H,$$

where (Positive definite i.e. $x^T \mathbf{B} x > 0$)

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & 0 & 0 & 0 \\ 0 & -2g^{00}\mathbb{I} & 0 & 0 \\ 0 & 0 & -2g^{00}\mathbb{I} & 0 \\ -(\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu) & 0 & 0 & -\mathbf{N} \end{pmatrix},$$

$$B^\mu = \begin{pmatrix} A^\mu & 0 & 0 & 0 \\ 0 & A^\mu & 0 & 0 \\ 0 & 0 & A^\mu & 0 \\ 0 & 0 & 0 & -N^\mu \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}^\star & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & \Pi & 0 \\ 0 & 0 & 0 & \mathbf{P}^\dagger \end{pmatrix}$$

and $H = (F_1, F_2, F_3, -F)^T$.

The Einstein equations (in the form of 1st order equation) become

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0^{0\mu} \\ \mathbf{u}_j^{0\mu} \\ \mathbf{u}^{0\mu} \end{pmatrix} = \frac{1}{\tau} \mathbf{A}\mathbf{P}^\star \begin{pmatrix} \mathbf{u}_0^{0\mu} \\ \mathbf{u}_j^{0\mu} \\ \mathbf{u}^{0\mu} \end{pmatrix} + F_1,$$

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0^{lm} \\ \mathbf{u}_j^{lm} \\ \mathbf{u}^{lm} \end{pmatrix} = \frac{1}{\tau}(-2g^{00})\Pi \begin{pmatrix} \mathbf{u}_0^{lm} \\ \mathbf{u}_j^{lm} \\ \mathbf{u}^{lm} \end{pmatrix} + F_2,$$

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_j \\ \mathbf{u} \end{pmatrix} = \frac{1}{\tau}(-2g^{00})\Pi \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_j \\ \mathbf{u} \end{pmatrix} + F_3,$$

where

$$A^0 = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & g^{ij} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix}, \quad A^k = \begin{pmatrix} -2g^{0k} & -g^{jk} & 0 \\ -g^{ik} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & \frac{3}{2}g^{li} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix},$$

$$\mathbf{P}^\star = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \delta_l^j & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 6\mathbf{u}^{0i}\mathbf{u}_i^{0\mu} + 4\mathbf{u}^{00}\mathbf{u}_0^{0\mu} - 4\mathbf{u}^{00}\mathbf{u}^{0\mu} + \mathfrak{H}^{0\mu} \\ 0 \\ 0 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 4\mathbf{u}^{00}\mathbf{u}_0^{ij} + \mathfrak{R}^{ij} \\ 0 \\ -g^{00}\mathbf{u}_0^{lm} \end{pmatrix}, \quad F_3 = \begin{pmatrix} 4\mathbf{u}^{00}\mathbf{u}_0 - 8(\mathbf{u}^{00})^2 + \mathfrak{F} \\ 0 \\ -g^{00}\mathbf{u}_0 \end{pmatrix}.$$

The dangerous terms in remainders F_ℓ: $\frac{1}{\tau^2}(\rho - \bar{\rho})$.

Pay the debts

- (Debt 3) Verification of $[\mathbf{P},\mathbf{B}] = \mathbf{PB} - \mathbf{BP} = 0$. Require

$$\mathbf{P}^\dagger(\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu) = (\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu)\mathbf{P}^\star$$

This implies

$$\begin{aligned}
&(\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu)\mathbf{P}^\star \\
&= \left(\frac{\tau\eta_{00}}{2\beta(\tau)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^l_r & 0 \end{pmatrix}\delta^0_\mu - \frac{\tau}{\beta(\tau)}(1-3\bar{s}^2)\begin{pmatrix} 0 & 0 & 0 \\ g_{rq} & 0 & g_{rq} \end{pmatrix}\delta^q_\mu\right)\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \delta^j_l & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
&= \frac{\tau\eta_{00}}{2\beta(\tau)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^j_r & 0 \end{pmatrix}\delta^0_\mu - \frac{\tau}{\beta(\tau)}(1-3\bar{s}^2)\begin{pmatrix} 0 & 0 & 0 \\ g_{rq} & 0 & g_{rq} \end{pmatrix}\delta^q_\mu \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \delta^i_r \end{pmatrix}\left(\frac{\tau\eta_{00}}{2\beta(\tau)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^j_i & 0 \end{pmatrix}\delta^0_\mu - \frac{\tau}{\beta(\tau)}(1-3\bar{s}^2)\begin{pmatrix} 0 & 0 & 0 \\ g_{iq} & 0 & g_{iq} \end{pmatrix}\delta^q_\mu\right) \\
&\qquad = \mathbf{P}^\dagger(\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu).
\end{aligned}$$

This answers why the factor $1\times\mathbf{u}^{0j} + 1\times\mathbf{u}^{0j}_0$ (GOOD!); $(1+6\bar{s}^2)\times\mathbf{u}^{0j} + 1\times\mathbf{u}^{0j}_0$ (BAD!)

Pay the debts

The choice of weight function $\beta(t)$ (debts 1 and 2):

- As we mentioned before, $\beta \sim (\tau^{\frac{2}{2n}}, \tau^{\frac{3}{2n}})$ to make sure the dangerous term $\frac{1}{\tau^2}(\rho - \bar{\rho})$ in F_ℓ of Einsten eqs. is regular in τ and also make sure ζ is bounded or with a small decay.
- Due to conditions $\frac{1}{\gamma_1}\mathbb{I} \leq \mathring{B}^0 \leq \frac{1}{\kappa}\mathring{\mathbf{B}} \leq \gamma_2\mathbb{I}$ and previous $\mathbf{P}^\dagger(\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu) = (\mathbf{E}_0\delta^0_\mu + \mathbf{E}_q\delta^q_\mu)\mathbf{P}^\star$, a entry in $\mathbf{N}$, $1 - 3\bar{s}^2 - \chi(\tau) = 1 - 3(\frac{\bar{\alpha}}{2n})^2 - \tau\partial_\tau \ln\beta > 0$, must be strictly positive. This condition leads to that for every $\varepsilon \in (0,1]$ take $n \in (\frac{3-\varepsilon}{2}, 3-\varepsilon] \subset (1,3)$ and $\beta(\tau) = C\tau^{(3-\varepsilon)/(2n)}$ and

$$0 < \bar{\alpha}(1) \leq 2n\sqrt{\frac{1}{3}\big(1 - \frac{3-\epsilon}{2n}\big)}.$$

Furthermore,

$$0 < \bar{\rho}(1) \leq \frac{1}{(4Kn(n+1))^n}\left(2n\sqrt{\frac{1}{3}\Big(1 - \frac{3-\epsilon}{2n}\Big)}\right)^{2n}.$$

Thank you
for your attention!