Cosmological Newtonian limits

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Einstein-Euler system with $\Lambda > 0$

Einstein-Euler equation with a positive cosmological constant $\Lambda > 0$ is given by

$$\tilde{G}^{\mu\nu} + \Lambda \tilde{g}^{\mu\nu} = \tilde{T}^{\mu\nu},$$
$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0,$$

where $\tilde{G}^{\mu\nu}$ is the Einstein tensor of the metric $\tilde{g} = \tilde{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$, and $\tilde{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\tilde{v}^\mu \tilde{v}^\nu + \bar{p}\tilde{g}^{\mu\nu}$ is the perfect fluid stress-energy tensor.
A linear equation of state of the form

$$\bar{p} = \epsilon^2 K \bar{\rho}, \qquad 0 < K \leq \frac{1}{3}.$$

$$\epsilon = \frac{v_T}{c},$$

where c is the speed of light and v_T is a characteristic speed associated to the fluid.

Conformal Poisson-Euler equations

The $\epsilon \searrow 0$ limit of the conformal Einstein-Euler equations on M are the *conformal cosmological Poisson-Euler equations*, which are defined by

$$\partial_t \mathring{\rho} + \sqrt{\frac{3}{\Lambda}}\partial_j \left(\mathring{\rho}\mathring{z}^j\right) = \frac{3(1-\mathring{\Omega})}{t}\mathring{\rho},$$

$$\sqrt{\frac{\Lambda}{3}}\mathring{\rho}\partial_t \mathring{z}^j + K\partial^j \mathring{\rho} + \mathring{\rho}\mathring{z}^i\partial_i \mathring{z}^j = \sqrt{\frac{\Lambda}{3}}\frac{1}{t}\mathring{\rho}\mathring{z}^j - \frac{1}{2}\frac{3}{\Lambda}\frac{t}{\mathring{E}}\mathring{\rho}\partial^j\mathring{\Phi},$$

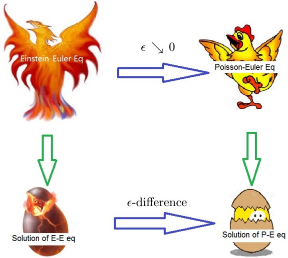
$$\Delta\mathring{\Phi} = \frac{\Lambda}{3}\frac{\mathring{E}^3}{t^3}\delta\mathring{\rho}.$$

where

$$\mathring{E}(t) = \left(\frac{C_0 - t^3}{C_0 - 1}\right)^{\frac{2}{3}} \quad \text{and} \quad \mathring{\Omega}(t) = \frac{2t^3}{t^3 - C_0},$$

$$C_0 = \frac{\sqrt{\Lambda + \mu(1)} + \sqrt{\Lambda}}{\sqrt{\Lambda + \mu(1)} - \sqrt{\Lambda}} > 1$$

Newtonian limits



- Field equation level (as $\epsilon \searrow 0$, GR field equations reduce to ones in NG.)
- Solution level (directly relate to physical phenomenon; this is what we concern about!)

Newtonian limits

Understanding the behavior of solutions to Einstein-matter equations in the limit $\epsilon \searrow 0$ is known as the Newtonian limit.

Analysis of Newtonian limits

We will refer to the global coordinates $(\bar{x}^\mu)$ on manifold M defined above as *relativistic coordinates*. In order to discuss the Newtonian limit and the sense in which solutions converge as $\epsilon \searrow 0$, we need to introduce the spatially rescaled coordinates (x^μ) defined by

$$t = \bar{x}^0 = x^0 \quad \text{and} \quad \bar{x}^i = \epsilon x^i, \qquad \epsilon > 0,$$

which we refer to as *Newtonian coordinates*.

The first key step for Newtonian limits is to identify the "right" variables which contain the right information of the orders of ϵ. Then by choosing suitable gauge, one can write Einstein-Euler equations, in terms of Newtonian coordinates, to the following singular symmetric hyperbolic equation

$$A^0(\epsilon, t, x, u)\partial_0 u + A^i(\epsilon, t, x, u)\partial_i u + \frac{1}{\epsilon}C^i\partial_i u = F(\epsilon, t, x, u)$$

where C^i are constant matrix.

The corresponding limiting equations of above singular hyperbolic equation is defined by

$$\begin{aligned} \mathring{A}^0(t,x,\mathring{u})\partial_0\mathring{u} + \mathring{A}^i(t,x,\mathring{u})\partial_i\mathring{u} + C^i\partial_i v =& \mathring{F}(t,x,\mathring{u}) \\ C^i\partial_i\mathring{u} =& 0 \end{aligned}$$

where, roughly speaking, $\mathring{A}^\mu := \lim_{\epsilon\searrow 0} A^\mu$ and $\mathring{F} := \lim_{\epsilon\searrow 0} F$. In fact, the Poisson-Euler equation in Newtonian gravity can be written in the form of above limiting equation. Therefore, one can regard u is the solution to Einstein-Euler equations and $\mathring{u}$ is the solution to Poisson-Euler equations. By Newtonian limits, we mean under suitable assumptions, we try to prove $\|u-\mathring{u}\|_{\text{some norm}} \leq C\epsilon$. This has a positive answer for *short time* region investigated by Browning, Klainerman, Schochet basing on some conditions. The main idea to handle this singular system in ϵ is to observe that in the energy estimate, $\frac{1}{\epsilon}\left\langle w, C^i\partial_i w\right\rangle \equiv 0$ due to C^i being a constant matrix. This will eliminate the worst singular term.

However, one difficulty we point out here is, usually, Einstein-Euler equations can not be written in the previous form directly, there will be a $1/\epsilon$ singular term appearing in the error term. To conquer this difficulty, we shift the unknown variables by some quantity ξ, then the $1/\epsilon$ singular term in the errors will be absorbed into $\frac{1}{\epsilon}C^i\partial_i w$ where $w = u - \xi$ and ξ. However, this shift will introduce the nonlocal term into the errors. Roughly speaking, this shifted component ξ is essentially related to the Newtonian potential and hence the nonlocal term is related to the poisson equations essentially.

FLRW metric

Friedmann-Lemaître-Robertson-Walker (FLRW) solution is an exact solution to Einstein field equation that represent a homogenous, isotropic, fluid filled universe undergoing accelerated expansion. Letting $(\bar{x}^i)$, $i = 1, 2, 3$, denote the standard coordinates on the $\mathbb{R}^3$ and $t = \bar{x}^0$ a time coordinate on the interval $(0, 1]$, the FLRW solutions on the manifold covered by $(\bar{x}^\mu)$

$$M_R := (0, 1] \times \mathbb{R}^3$$

are defined by $(t = e^{-\sqrt{\frac{\Lambda}{3}}\tau})$

$$\tilde{h}(t) = -\frac{3}{\Lambda t^2} dtdt + a(t)^2 \delta_{ij} d\bar{x}^i d\bar{x}^j,$$

$$\tilde{v}_H(t) = -t\sqrt{\frac{\Lambda}{3}}\partial_t,$$

$$\mu(t) = \frac{\mu(1)}{a(t)^{3(1+\epsilon^2 K)}},$$

Fundamental Question and Motivations

Question

On what space and time scales Newtonian cosmological simulations can be trusted to approximate relativistic cosmologies?

Motivations

1. Dark energy, cosmological averaging, backreactions
2. Answer old hidden assumption in physics, post-Newtonian expansion
3. Approximate GR using NG

4 step-Answers

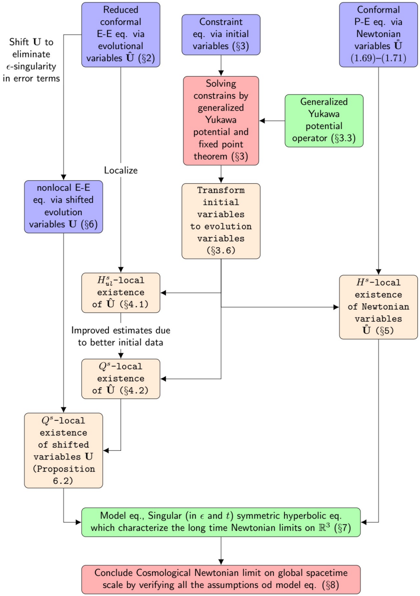
1. "Todd A. Oliynyk: Cosmological post-Newtonian expansions to arbitrary order (2010)" answers this question on small space-time scales;
2. "Todd A. Oliynyk: The Newtonian limit on cosmological scales (2015)" answers this on large spatial scales but small temporal scale;
3. "C.L. and Todd A. Oliynyk, Cosmological newtonian limits on long time scales (2017)" answers this on large temporal scale but small spatial scales;
4. "C.L. and Todd A. Oliynyk, Cosmological newtonian limits on large scales (2017)" answers this on large space-time scales.

Answers on large scales (rough expression)

We answer this question under a small initial data condition. Informally, we construct initial data sets that solve the constraint equations and represent initial conditions relevant to realistic cosmologies and establish the existence of 1-parameter families of ϵ-dependent solutions to Einstein-Euler system with positive cosmological constant $\Lambda > 0$ that:

1. are defined for $\epsilon \in (0, \epsilon_0)$ for some fixed constant $\epsilon_0 > 0$,
2. exist globally on $(t, x^i) \in [0, +\infty) \times \mathbb{R}^3$,
3. converge, in a suitable sense, as $\epsilon \searrow 0$ to solutions of the cosmological Poison-Euler equations of Newtonian gravity,
4. are small, non-linear perturbations of the FLRW fluid solutions.

A glance at the main proof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Evolutions of Poisson potential

Shift the following evolution Poisson potential to eliminate the ϵ-singularity in errors

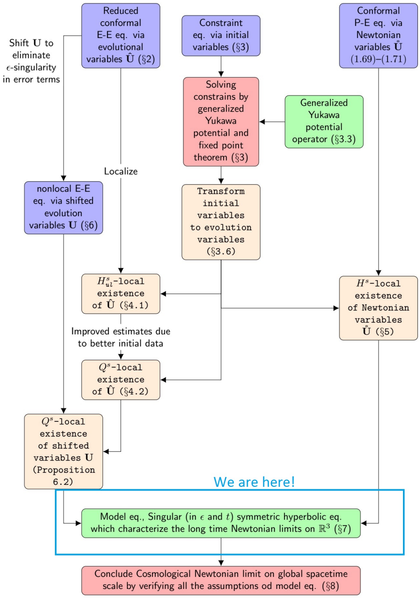
$$\Phi^{\mu}_{k} := \frac{\Lambda}{3}\frac{E^3}{t^3}\delta^{\mu}_{0}\partial_k(\Delta - \epsilon^2\beta)^{-1}\big(E^{-3}\sqrt{|\bar{g}|}\varrho\bar{v}^0 - \mu^{\frac{1}{1+\epsilon^2 K}}\big). \tag{1}$$

By 1st Euler equation (continuity equation), we can derive a evolution eq of Φ^{μ}_{k} (ODE),

$$\partial_t\Phi^{\mu}_{k} = -\frac{\Lambda}{3}(\Delta - \epsilon^2\beta)^{-1}\partial_k\partial_l\big(\delta^{\mu}_{0}\sqrt{|\bar{g}|}e^{\zeta}z^l\big). \tag{2}$$

then obtain nice estimate of it

The evolution equations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Conformal Einstein equations

Replace the physical (inverse) metric $\tilde{g}^{\mu\nu}$ and fluid four-velocity $\tilde{v}^{\mu}$ by the conformally rescaled versions defined by

$$\bar{g}^{\mu\nu} = e^{2\Psi}\tilde{g}^{\mu\nu} \quad \text{and} \quad \bar{v}^{\mu} = e^{\Psi}\tilde{v}^{\mu}.$$

Then the conformal Einstein equation is

$$\begin{aligned} &\bar{G}^{\mu\nu} = \bar{T}^{\mu\nu} \\ :=&e^{4\Psi}\tilde{T}^{\mu\nu} - e^{2\Psi}\Lambda\bar{g}^{\mu\nu} + 2(\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}\Psi - \bar{\nabla}^{\mu}\Psi\bar{\nabla}^{\nu}\Psi) - (2\bar{\Box}\Psi + |\bar{\nabla}\Psi|^2_{\bar{g}})\bar{g}^{\mu\nu}, \end{aligned}$$

where $\bar{\nabla}_{\mu}$ and $\bar{G}_{\mu\nu}$ are the covariant derivative and Einstein tensor of $\bar{g}_{\mu\nu}$, respectively, $\bar{\Box} = \bar{\nabla}^{\mu}\bar{\nabla}_{\mu}$, and $|\bar{\nabla}\Psi|^2_{\bar{g}} = \bar{g}^{\mu\nu}\bar{\nabla}_{\mu}\Psi\bar{\nabla}_{\nu}\Psi$.

Furthermore, another representation of *the conformal Einstein equation* is

$$\begin{aligned}\bar{R}^{\mu\nu} =& 2\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}\Psi - 2\bar{\nabla}^{\mu}\Psi\bar{\nabla}^{\nu}\Psi + \Big[\bar{\Box}\Psi + 2|\bar{\nabla}\Psi|^2 \\ &+ \left(\frac{1-\epsilon^2 K}{2}\bar{\rho} + \Lambda\right) e^{2\Psi}\Big]\bar{g}^{\mu\nu} + e^{2\Psi}(1+\epsilon^2 K)\bar{\rho}\bar{v}^{\mu}\bar{v}^{\nu}.\end{aligned}$$

where Ricci tensor can be expressed as

$$\bar{R}^{\mu\nu} = \frac{1}{2}\bar{g}^{\lambda\sigma}\bar{\nabla}_{\lambda}\bar{\nabla}_{\sigma}\bar{g}^{\mu\nu} + \bar{\nabla}^{(\mu}\Gamma^{\nu)} + \bar{\mathcal{R}}^{\mu\nu} + \bar{P}^{\mu\nu} + \bar{Q}^{\mu\nu}.$$

Conformal singularized, reduced Einstein-Euler equations

1. Long time problems of singular symmetric hyperbolic equations with damped terms $\rightsquigarrow$ Short time existence problem of a singular symmetric hyperbolic system with respect to $1/\epsilon$ and $1/t$:

$$B^0\partial_0 u + B^i\partial_i u + \frac{1}{\epsilon}C^i\partial_i u = \frac{1}{t}\mathcal{B}\mathbb{P}u + H + G$$

2. $t \to +\infty$ problem $\rightsquigarrow$ $t \to 0$ singular problem;
$\tau \in (0, +\infty) \rightsquigarrow t \in [T_0, 0)$

Ricci tensor can be expressed as

$$\bar{R}^{\mu\nu} = \frac{1}{2}\bar{g}^{\lambda\sigma}\bar{\bar{\nabla}}_\lambda\bar{\bar{\nabla}}_\sigma\bar{g}^{\mu\nu} + \bar{\nabla}^{(\mu}\bar{X}^{\nu)} + \bar{\mathcal{R}}^{\mu\nu} + \bar{P}^{\mu\nu} + \bar{Q}^{\mu\nu}.$$

Using suitable variables, rescaling spatial coordinates $\bar{x}^i = \epsilon x^i$, conformal factors $\Psi = -\ln t$ and generalized wave gauge

$$\bar{Z}^\mu = \bar{X}^\mu + \bar{Y}^\mu = 0,$$

where

$$\bar{X}^\mu := \Gamma^\mu = \bar{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = -\bar{\bar{\nabla}}_\nu\bar{g}^{\mu\nu} + \frac{1}{2}\bar{g}^{\mu\sigma}\bar{g}_{\alpha\beta}\bar{\bar{\nabla}}_\sigma\bar{g}^{\alpha\beta}$$

and

$$\bar{Y}^\mu := -2\bar{\nabla}^\mu\Psi + 2\bar{\bar{\nabla}}^\mu\Psi = \frac{2}{t}\left(\bar{g}^{\mu 0} + \frac{\Lambda}{3}\delta^\mu_0\right).$$

$$u^{0\mu} = \frac{1}{\epsilon}\frac{\underline{\bar{g}^{0\mu}} - \bar{h}^{0\mu}}{2t}, \tag{1.40}$$

$$u^{0\mu}_0 = \frac{1}{\epsilon}\left(\delta^0_\nu \underline{\bar{\nabla}_0 \bar{g}^{\mu\nu}} - \frac{3(\underline{\bar{g}^{0\mu}} - \bar{h}^{0\mu})}{2t}\right), \tag{1.41}$$

$$u^{0\mu}_i = \frac{1}{\epsilon}\delta^0_\nu \underline{\bar{\nabla}_i \bar{g}^{\mu\nu}}, \tag{1.42}$$

$$u^{ij} = \frac{1}{\epsilon}(\underline{\bar{\mathfrak{g}}^{ij}} - \bar{h}^{ij}), \tag{1.43}$$

$$u^{ij}_\mu = \frac{1}{\epsilon}\delta^i_\sigma \delta^j_\nu \underline{\bar{\nabla}_\mu (\alpha^{-1}\bar{g}^{\sigma\nu} - \bar{h}^{\sigma\nu})}, \tag{1.44}$$

$$u = \frac{1}{\epsilon}\underline{\bar{\mathfrak{q}}}, \tag{1.45}$$

$$u_\mu = \frac{1}{\epsilon}\left(\underline{\delta^0_\sigma \delta^0_\nu \bar{\nabla}_\mu (\bar{g}^{\sigma\nu} - \bar{h}^{\sigma\nu})} - \frac{\Lambda}{3}\underline{\bar{\nabla}_\mu \ln \alpha}\right), \tag{1.46}$$

$$z_i = \frac{1}{\epsilon}\underline{\bar{v}_i}, \tag{1.47}$$

$$\zeta = \frac{1}{1+\epsilon^2 K}\ln\left(t^{-3(1+\epsilon^2 K)}\underline{\bar{\rho}}\right), \tag{1.48}$$

and

$$\delta\zeta = \zeta - \zeta_H \tag{1.49}$$

where

$$\bar{\mathfrak{g}}^{ij} = \alpha^{-1}\bar{g}^{ij}, \qquad \alpha := (\det \bar{g}^{kl})^{\frac{1}{3}}/(\det \bar{h}^{kl})^{\frac{1}{3}} = E^2(\det \breve{g}_{ij})^{-\frac{1}{3}} = E^2(\det \bar{g}^{kl})^{\frac{1}{3}},$$

$$\breve{g}_{ij} = (\bar{g}^{ij})^{-1}, \tag{1.50}$$

$$\bar{\mathfrak{q}} = \bar{g}^{00} - \bar{h}^{00} - \frac{\Lambda}{3}\ln \alpha, \tag{1.51}$$

$$\zeta_H(t) = \frac{1}{1+\epsilon^2 K}\ln\left(t^{-3(1+\epsilon^2 K)}\mu(t)\right) \tag{1.52}$$

We are able to write Einstein-Euler equations as the following Model equation

$$B^0\partial_0 u + B^i\partial_i u + \frac{1}{\epsilon}C^i\partial_i u = \frac{1}{t}\mathcal{B}\mathbb{P}u + H + G$$

in $[T_0, T_1) \times \mathbb{R}^3$, $T_1 < 0$.
Main observation: $\frac{1}{t}\mathcal{B}\mathbb{P}u$ has the "right" sign due to positive cosmological constant.

Then main aim is to analyze this system in $Q^s(\mathbb{R}^3) = R^s(\mathbb{R}^3)$ (due to initial data) spaces, where

$$Q^s(\mathbb{R}^3) := \left\{ u \in W^{s-1,6}(\mathbb{R}^3) \bigcap W^{s-2,\infty}(\mathbb{R}^3) \mid Du \in H^{s-1}(\mathbb{R}^3) \right\}$$
$$R^s(\mathbb{R}^3) := \left\{ u \in L^6(\mathbb{R}^3) \mid Du \in H^{s-1}(\mathbb{R}^3) \right\}$$

with norm

$$\|u\|_{Q^s(\mathbb{R}^3)} = \|Du\|_{H^{s-1}(\mathbb{R}^3)} + \|u\|_{W^{s-1,6}(\mathbb{R}^3)} + \|u\|_{W^{s-2,\infty}(\mathbb{R}^3)},$$
$$\|u\|_{R^s(\mathbb{R}^3)} = \|Du\|_{H^{s-1}(\mathbb{R}^3)} + \|u\|_{L^6(\mathbb{R}^3)}$$

Basic idea: energy estimates

Aim: 1. Long time solution by continuation principle require $\|u\|_{W^{1,\infty}} \leq \infty$.
2. $\|u - \mathring{u}\|_{L^\infty([T_0,0),Q^{s-1})} \lesssim \epsilon$.

Step 1: Analyze model equation, by energy estimate,

$$B^0\partial_0 u + B^i\partial_i u + \frac{1}{\epsilon}C^i\partial_i u = \frac{1}{t}\mathcal{B}\mathbb{P}u + H + G$$

under certain structural assumptions with small initial data.
Step 2: Writing E-E eq (e.g variables w), P-E eq (e.g variables $\mathring{w}$) to model equation. Then some operations on E-E eq and P-E eq to derive eqs of $\mathbb{P}_1\mathring{w}$, $t\partial_t\mathring{w}$ and $w - \mathring{w}$ all into model eq. Using theorem of Step 1, we conclude Aim 1 and 2 together.

Toy Model

let's focus on a simple linear model to convey the spirit how to eliminate the singular terms.

$$\partial_t U + \frac{1}{\epsilon} a^i \partial_i U = \frac{1}{t} U + aU \quad \text{where } \|U\| = \langle U, U \rangle^{\frac{1}{2}}$$

for $t \in [-1, 0)$, $\epsilon \in (0, \epsilon_0)$.

Then by simple energy estimates

$$\partial_t \|U\|^2 = \frac{2}{t}\|U\|^2 + 2a\|U\|^2 \tag{3}$$

provided

$$\frac{1}{\epsilon}\langle U, a^i \partial_i U\rangle = 0. \tag{4}$$

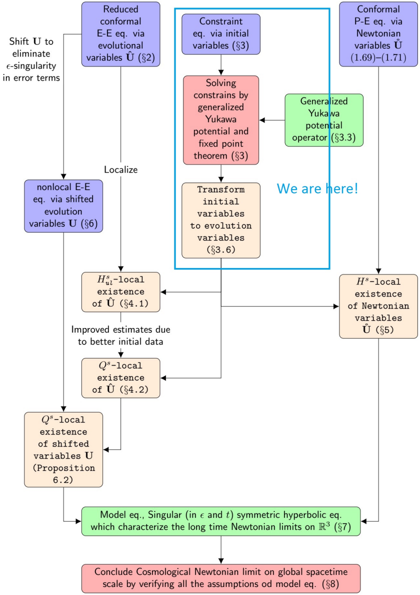
This condition (4), in fact, holds for all of our proof due to the manifolds of integration are $\mathbb{T}^3$ and $\mathbb{R}^3$ respectively. Then we conclude that

$$\partial_t \left(\|U\|^2 + \int_{T_0}^{t} -\frac{2}{s}\|U\|^2 ds \right) \lesssim \|U\|^2 + \int_{T_0}^{t} -\frac{2}{s}\|U\|^2 ds. \tag{5}$$

Then applying Grönwall's inequality leads to the boundness of a new energy

$$\|U\|^2 + \int_{T_0}^{t} -\frac{2}{s}\|U\|^2 ds \quad \text{(Note that } s < 0\text{).} \tag{6}$$

The constraint equations



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Constraints

The complete set of constraints on Σ_{T_0} are:

$$\begin{aligned} (\bar{G}^{0\mu} - \bar{T}^{0\mu})|_{t=T_0} &= 0 \quad \text{(Gravitational Constraints)},\\ \bar{Z}^{\mu}|_{t=T_0} &= 0 \quad \text{(Gauge constraint)}\\ (\bar{v}^{\mu}\bar{v}_{\mu} + 1)|_{t=T_0} &= 0 \quad \text{(Velocity Normalization)}. \end{aligned}$$

Gauge constraints leads to an identity like

$$\partial_0 u^{0\mu} = -\frac{1}{\epsilon}\partial_k u^{k\mu} + \text{remaiders (regular in } \epsilon)$$

Using this to replace all the time derivatives in nonlinear wave operator (from gravitational constraints) and move all high order ϵ terms into remainders, we derive a eq

$$\begin{pmatrix} \Delta - \epsilon^2 a & -\epsilon b \partial_j \\ -\epsilon d \partial^j & \Delta - \epsilon^2 c \end{pmatrix} \begin{pmatrix} \phi \\ \psi^j \end{pmatrix} = \begin{pmatrix} f(\epsilon, \phi, \psi^j) \\ g^j(\epsilon, \phi, \psi^j) \end{pmatrix}. \tag{7}$$

where $\phi = u^{00}$, $\psi^j = u^{0j}$ and constants $a > 0$, $b < 0$, $c > 0$ and $d > 0$.

Aim: construct a strict contraction.
Direct idea is to inverse the operator matrix, but only true for $(a + bd + c)^2 - 4ac \geq 0$. This is not always true for our case.

Transformed into

$$\begin{pmatrix} \Delta - \epsilon^2(\mathtt{a} + \mathtt{bd}) & -\epsilon \mathtt{b}\partial_j \\ 0 & \Delta - \epsilon^2 \mathtt{c} \end{pmatrix} \begin{pmatrix} \phi \\ \vartheta^j \end{pmatrix} = \begin{pmatrix} \tilde{f}(\epsilon, \phi, \vartheta^k, \breve{\xi}) \\ \tilde{g}^j(\epsilon, \phi, \vartheta^k, \breve{\xi}) \end{pmatrix} \tag{8}$$

where we have introduced a new variable

$$\vartheta^j = \psi^j - \epsilon \mathtt{d}\partial^j(\Delta - \epsilon^2 \mathtt{c})^{-1}\phi \tag{9}$$

Then, we can inverse the operator matrix

$$\begin{pmatrix} \Delta - \epsilon^2(\mathtt{a} + \mathtt{bd}) & \epsilon \mathtt{b}\partial_j \\ 0 & \Delta - \epsilon^2 \mathtt{c} \end{pmatrix}^{-1} = \begin{pmatrix} (\Delta - \epsilon^2(\mathtt{a} + \mathtt{bd}))^{-1} & \epsilon \mathtt{b}\partial_j(\Delta - \epsilon^2 \mathtt{c})^{-1}(\Delta - \epsilon^2(\mathtt{a} + \mathtt{bd}))^{-1} \\ 0 & (\Delta - \epsilon^2 \mathtt{c})^{-1} \end{pmatrix}. \tag{10}$$

This requires some detailed investigation of operator $(\epsilon^2 - \Delta)^{-1}$, which we call it a *generalized Yukawa potential operator*.

Banach's fixed point theorem to conclude the existence of φ and ψ^j

Contraction:

$$\begin{aligned}\grave{\phi} =&\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\tilde{f}(\epsilon, \acute{\phi}, \acute{\vartheta}^k, \breve{\xi}) \\ &+ \epsilon\mathtt{b}\partial_j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\tilde{g}^j(\epsilon, \acute{\phi}, \acute{\vartheta}^k, \breve{\xi}), \\ \grave{\vartheta}^k =&(\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^k(\epsilon, \acute{\phi}, \acute{\vartheta}^j, \breve{\xi}).\end{aligned}$$

Using generalized Yukawa operator as a tool with very delicate structure of $\tilde{f}$ and $\tilde{g}^j$, we know this mapping is a contraction on R^{s+1}. Then apply *Banach's fix point theorem*.

Generalized Yukawa potential operator

Definition

Let $s \in \mathbb{R}$ such that $0 < s < \infty$ and $\kappa \geq 0$ is a constant. The generalized Yukawa potential operator of order s is defined as $(\kappa^2 - \Delta)^{-\frac{s}{2}}$. This operator acts on function f as follows:

$$(\kappa^2 - \Delta)^{-\frac{s}{2}}(f) = (\widehat{\mathcal{Y}}_{s,\kappa}\widehat{f})^{\vee} = \mathcal{Y}_{s,\kappa} * f, \tag{11}$$

where

$$\mathcal{Y}_{s,\kappa}(x) = \left((\kappa^2 + 4\pi^2|\xi|^2)^{-\frac{s}{2}}\right)^{\vee}(x). \tag{12}$$

Main properties of GYPO

For all $0 < s < \infty$ and $\kappa > 0$, we have

$$\|\kappa^s(-\Delta+\kappa^2)^{-\frac{s}{2}}(f)\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}, \tag{13}$$

$$\|\partial_j(-\Delta+\kappa^2)^{-\frac{1}{2}}f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{14}$$

and if

$$\frac{1}{p}-\frac{1}{q}=\frac{s}{n}. \tag{15}$$

then

$$\|(\kappa^2-\Delta)^{-\frac{s}{2}}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{16}$$

Lemma: estimates of the RHS

Suppose $s \in \mathbb{Z}_{\geq 3}$, $0 < \epsilon < \epsilon_0$, $\lambda \in \mathbb{R}_{>0}$, and F is defined by

$$F = \epsilon^4 \mathtt{H}_1(\epsilon, f_1, f_2) + \epsilon \partial_i \partial_j f_3 + f_4 + \epsilon^3 \mathtt{H}_5(\epsilon, f_5, \partial_i \partial_j f_6) + \epsilon^3 \mathtt{H}_7(\epsilon, f_7, \partial_i f_8) + \epsilon^3 \mathtt{H}_0(\epsilon, f_0, f_8) + \epsilon^3 f_9 + \epsilon^2 f_{10} + \epsilon \partial_i f_{11} + \epsilon f_{12}$$

where $f_1, f_2, f_3, f_5, f_6, f_7, f_9 \in R^{s+1}(\mathbb{R}^3)$, $f_4, f_{12} \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3)$, $f_0 \in R^s(\mathbb{R}^3)$, $f_8, f_{10}, f_{11} \in H^s(\mathbb{R}^3)$, and the maps $\mathtt{H}_\ell(\epsilon, u, v)$, $\ell = 0, 1, 5, 7$, are smooth, vanish to first order in u, and are linear in v. Then $(\epsilon^2 \lambda - \Delta)^{-1} F \in R^{s+1}$ and

$$\|(\epsilon^2 \lambda - \Delta)^{-1} F\|_{R^{s+1}} \leq C_0 \Big[\epsilon^2 \|f_1\|_{R^{s+1}} \|f_2\|_{R^{s+1}} + \epsilon \|f_3\|_{R^{s+1}} + \|f_4\|_{L^{\frac{6}{5}} \cap K^s} + \epsilon \|f_5\|_{R^{s+1}} \|f_6\|_{R^{s+1}} + \epsilon \|f_9\|_{R^{s+1}} + \epsilon (\|f_7\|_{R^{s+1}} + \|f_0\|_{R^s}) \|f_8\|_{H^s} + \epsilon \|f_{10}\|_{H^s} + \epsilon \|f_{11}\|_{H^s} + \epsilon \|f_{12}\|_{L^{\frac{6}{5}} \cap K^s} \Big]$$

where $C_0 = C_0\big(\|f_0\|_{R^s}, \|f_1\|_{R^{s+1}}, \|f_2\|_{R^{s+1}}, \|f_5\|_{R^{s+1}}, \|f_6\|_{R^{s+1}}, \|f_7\|_{R^{s+1}}, \|f_8\|_{H^s}\big)$.
Furthermore, if $f_{10} = \mathtt{G}(f) g$, where $f \in K^s(\mathbb{R}^3)$, $g \in H^s(\mathbb{R}^3)$ and $\mathtt{G}(u)$ is smooth, then

$$\|f_{10}\|_{H^s} \leq C(\|f\|_{K^s}) \|g\|_{H^s},$$

and, in the case $\mathtt{G}(u)$ also vanishes to first order in u,

$$\|f_{10}\|_{H^s} \leq C(\|f\|_{K^s}) \|f\|_{K^s} \|g\|_{H^s}.$$

Rough result: Free initial data: $\breve{\mathfrak{u}}^{ij} \in R^{s+1}(\mathbb{R}^3, \mathbb{S}_3)$, $\breve{\mathfrak{u}}^{ij}_0 \in H^s(\mathbb{R}^3, \mathbb{S}_3)$, $\delta\breve{\rho} \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R})$ and $\breve{z}^j \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R}^3)$ determine $\breve{\mathfrak{u}}^{0\mu} \in R^{s+1}(\mathbb{R}^3, \mathbb{R}^4)$ and $\breve{\mathfrak{u}}^{0\mu}_0 \in R^s(\mathbb{R}^3, \mathbb{R}^4)$. Furthermore, the complete initial data set.

Glue data

$$\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^{N} \delta\breve{\rho}_{\lambda}\left(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\right) \quad \text{and} \quad \breve{z}^j_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^{N} \breve{z}^j_{\lambda}\left(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\right),$$

Thank you
for your attention!

Theorem 3.16. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $r > 0$, $\epsilon_1 > 0$, $\vec{\mathbf{y}} = (\mathbf{y}_1, \cdots, \mathbf{y}_N) \in \mathbb{R}^{3N}$, $\breve{\mathfrak{u}}^{ij}_\epsilon \in R^{s+1}(\mathbb{R}^3, \mathbb{S}_3)$ and $\breve{\mathfrak{u}}^{ij}_{0,\epsilon} \in R^s(\mathbb{R}^3, \mathbb{S}_3) \cap L^2(\mathbb{R}^3, \mathbb{S}_3)$ for $\epsilon \in (0, \epsilon_1)$, $\delta\breve{\rho}_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R})$ and $\breve{z}^j_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R}^3)$ for $\lambda = 1, \cdots, N$, $\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}$ and $\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}$ are defined by (3.115) and $\mu(1)$ satisfies (3.64). Then there exists a constant $\epsilon_0 \in (0, \epsilon_1)$ such that if the free initial data satisfies*

$$\|\breve{\xi}_\epsilon\|_s := \|\breve{\mathfrak{u}}^{ij}_\epsilon\|_{R^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s} + \sum_{\lambda=1}^N \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}} \cap K^s} + \sum_{\lambda=1}^N \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}} \cap K^s} \leq r, \quad 0 < \epsilon < \epsilon_0,$$

then there exists a family $(\epsilon, \vec{\mathbf{y}})$-dependent maps

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}, u_{\epsilon,\vec{\mathbf{y}}}, u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}, u_{\gamma,\epsilon,\vec{\mathbf{y}}}, z_{j,\epsilon,\vec{\mathbf{y}}}, \delta\zeta_{\epsilon,\vec{\mathbf{y}}}\}|_\Sigma, \quad (\epsilon, \vec{\mathbf{y}}) \in (0, \epsilon_0) \times \mathbb{R}^{3N},$$

such that $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma \in X^s(\mathbb{R}^3)$, $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ determines a solution of the constraint equations (3.3)-(3.5), and the components of $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ can be expressed as

$$u^{0\mu}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon \mathcal{S}^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.117}$$

$$u_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon^2 \frac{2\Lambda}{9} E^2(1) \breve{\mathfrak{u}}^{ij}_\epsilon \delta_{ij} + \epsilon^3 \mathcal{S}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.118}$$

$$u^{ij}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon^2 \left(\breve{\mathfrak{u}}^{ij}_\epsilon - \frac{1}{3} \breve{\mathfrak{u}}^{kl}_\epsilon \delta_{kl} \delta^{ij} \right) + \epsilon^3 \mathcal{S}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.119}$$

$$z_{j,\epsilon,\vec{\mathbf{y}}}|_\Sigma = E^2(1) \delta_{kl} \breve{z}^k_{\epsilon,\vec{\mathbf{y}}} + \epsilon \mathcal{R}_j(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.120}$$

$$\delta\zeta_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \frac{1}{1+\epsilon^2 K} \ln \left(1 + \frac{\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}}{\mu(1)} \right), \tag{3.121}$$

$$u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \frac{\Lambda}{3} E^2(1) \delta^\mu_0 \partial_i \Delta^{-1} \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}} + \epsilon \mathcal{S}^\mu_i(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.122}$$

$$u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon \mathcal{S}^\mu_0(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.123}$$

$$u_{\gamma,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon \mathcal{S}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.124}$$

and

$$u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon \mathcal{S}^{ij}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.125}$$

where the remainders are bounded by

$$\begin{aligned} &\|\mathcal{S}^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} \\ &\quad + \|\mathcal{R}_j(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^\mu_i(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^\mu_0(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} \\ &\quad + \|\mathcal{S}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^{ij}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} \lesssim \|\breve{\xi}\|_s \end{aligned}$$

for all $(\epsilon, \vec{\mathbf{y}}) \in (0, \epsilon_0) \times \mathbb{R}^{3N}$. Moreover, the components of $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ satisfy the uniform bounds

$$\|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^{s+1}} + \|u_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^{s+1}} + \|u^{0k}_{i,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|u_{\mu,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|u^{ij}_{\mu,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} \lesssim \epsilon \|\breve{\xi}_\epsilon\|_s$$

and

$$\|u^{00}_{i,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|\breve{z}_{j,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|\delta\zeta_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{L^{\frac{6}{5}} \cap K^s} \lesssim \|\breve{\xi}_\epsilon\|_s$$

for all $(\epsilon, \vec{\mathbf{y}}) \in (0, \epsilon_0) \times \mathbb{R}^{3N}$.

Main Theorem

Suppose $s \in \mathbb{Z}_{\geq 3}$, $0 < K \leq \frac{1}{3}$, $\Lambda > 0$, $\mu(1) > 0$, $r > 0$, $\vec{\mathbf{y}} = (\mathbf{y}_1, \cdots, \mathbf{y}_N) \in \mathbb{R}^{3N}$ and the free initial data $\{\breve{\mathfrak{u}}^{ij}_\epsilon, \breve{\mathfrak{u}}^{ij}_{0,\epsilon}, \delta\breve{\rho}_\lambda, \breve{z}^i_\lambda\}$ is chosen so that $\breve{\mathfrak{u}}^{ij}_\epsilon \in R^{s+1}(\mathbb{R}^3, \mathbb{S}_3)$, $\breve{\mathfrak{u}}^{ij}_{0,\epsilon} \in H^s(\mathbb{R}^3, \mathbb{S}_3)$, $\delta\breve{\rho}_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R})$ and $\breve{z}^j_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R}^3)$ for $\lambda = 1, \cdots, N$. $\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}$ and $\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}$ are defined by

$$\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^{N} \delta\breve{\rho}_\lambda\left(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\right) \quad \text{and} \quad \breve{z}^j_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^{N} \breve{z}^j_\lambda\left(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\right), \tag{17}$$

Then for $r > 0$ chosen small enough such that

$$\|\breve{\mathfrak{u}}^{ij}_\epsilon\|_{R^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s} + \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}} \cap K^s} + \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}} \cap K^s} \leq r. \tag{18}$$

Then there exists an small constant $\epsilon_0 = \epsilon_0(r) > 0$ and maps $\breve{u}^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^{s+1}(\mathbb{R}^3, \mathbb{S}_4)$, $\breve{u}_{\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^{s+1}(\mathbb{R}^3)$, $\breve{u}^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^s(\mathbb{R}^3, \mathbb{S}_4)$, $\breve{u}_{0,\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^s(\mathbb{R}^3)$, $\breve{z}_{i,\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^s(\mathbb{R}^3, \mathbb{R}^3)$, and $\delta\breve{\zeta}_{\epsilon,\vec{\mathbf{y}}} : (0, \epsilon_0) \times \left(L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3)\right) \to L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3)$, such that

$$\begin{aligned}
u^{0\mu}_{\epsilon,\vec{\mathbf{y}}}|_{t=1} =& \breve{u}^{0\mu}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_{\epsilon}, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^{l}_{\epsilon,\vec{\mathbf{y}}}) \\
=& \frac{\epsilon\Lambda}{12}\delta^{\mu}_{0}\Big[2(\Delta-\epsilon^2 a)^{-1}\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}} + \epsilon b(\Delta-\epsilon^2 c)^{-1}\partial_j\Big(-\epsilon(\Delta-\epsilon^2 c)^{-1}\partial_i\breve{\mathfrak{u}}^{ji}_{0,\epsilon} \\
& + 2\epsilon d\partial^j(\Delta-\epsilon^2 c)^{-1}(\Delta-\epsilon^2 a)^{-1}\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\Big)\Big] + \frac{\epsilon\Lambda}{6}\delta^{\mu}_{j}\Big(-\epsilon(\Delta-\epsilon^2 c)^{-1}\partial_i\breve{\mathfrak{u}}^{ji}_{0,\epsilon} \\
& + 2\epsilon d\partial^j(\Delta-\epsilon^2 c)^{-1}(\Delta-\epsilon^2 a)^{-1}\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\Big) + O(\epsilon^2), \qquad (19)
\end{aligned}$$

$$u_{\epsilon,\vec{\mathbf{y}}}|_{t=1} = \breve{u}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_{\epsilon}, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^{l}_{\epsilon,\vec{\mathbf{y}}}) = \epsilon^2\frac{2\Lambda}{9}E^2(1)\breve{\mathfrak{u}}^{ij}_{\epsilon}\delta_{ij} + O(\epsilon^3), \qquad (20)$$

$$u^{ij}_{\epsilon,\vec{\mathbf{y}}}|_{t=1} = \breve{u}^{ij}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_{\epsilon}, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^{l}_{\epsilon,\vec{\mathbf{y}}}) = \epsilon^2\left(\breve{\mathfrak{u}}^{ij}_{\epsilon} - \frac{1}{3}\breve{\mathfrak{u}}^{kl}_{\epsilon}\delta_{kl}\delta^{ij}\right) + O(\epsilon^3), \qquad (21)$$

$$z_{j,\epsilon,\vec{\mathbf{y}}}|_{t=1} = \breve{z}_{j,\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_{\epsilon}, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^{l}_{\epsilon,\vec{\mathbf{y}}}) = E^2(1)\delta_{kl}\breve{z}^{k}_{\epsilon,\vec{\mathbf{y}}} + O(\epsilon), \qquad (22)$$

$$\delta\zeta_{\epsilon,\vec{\mathbf{y}}}|_{t=1} = \delta\breve{\zeta}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}) = \frac{1}{1+\epsilon^2 K}\ln\left(1+\frac{\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}}{\mu(1)}\right), \qquad (23)$$

$$u^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}}|_{t=1} = \breve{u}^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_{\epsilon}, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^{l}_{\epsilon,\vec{\mathbf{y}}}) = O(\epsilon), \qquad (24)$$

and

$$u_{0,\epsilon,\vec{\mathbf{y}}}|_{t=1} = \breve{u}_{0,\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_{\epsilon}, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^{l}_{\epsilon,\vec{\mathbf{y}}}) = O(\epsilon), \qquad (25)$$

satisfying the estimate

$$\|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}|_{t=1}\|_{R^{s+1}} + \|u_{\epsilon,\vec{\mathbf{y}}}|_{t=1}\|_{R^{s+1}} + \|u_{0,\epsilon,\vec{\mathbf{y}}}|_{t=1}\|_{R^s} + \|u^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}}|_{t=1}\|_{R^s} + \|\delta\zeta_{\epsilon,\vec{\mathbf{y}}}|_{t=1}\|_{L^{\frac{6}{5}}\cap K^s}$$
$$+ \|z_{j,\epsilon,\vec{\mathbf{y}}}|_{t=1}\|_{R^s} \lesssim \|\breve{\mathfrak{u}}^{ij}_{\epsilon}\|_{R^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s} + \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}}\cap K^s} + \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}}\cap K^s}, \tag{26}$$

where

$$a = \frac{\Lambda}{3}\big(7 - 6\Omega(1)\big), \qquad b = \frac{\Lambda}{3}\big(\Omega(1) - 1\big), \tag{27}$$
$$c = 2\Lambda(1+\epsilon^2 K)\big(\Omega(1) - 2\big)\Omega(1) \quad \text{and} \quad d = -2\Omega(1), \tag{28}$$

determine a solution of the gravitational and gauge constraint equations. Furthermore, there exists a $\sigma > 0$, such that if

$$\|\breve{\mathfrak{u}}^{ij}_{\epsilon}\|_{R^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s} + \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}}\cap K^s} + \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}}\cap K^s} \leq \sigma,$$

then there exist maps

$$u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], Q^s(\mathbb{R}^3,\mathbb{S}_4)) \cap C^1((0,1], Q^{s-1}(\mathbb{R}^3,\mathbb{S}_4)),$$
$$u^{\mu\nu}_{\gamma,\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], Q^s(\mathbb{R}^3,\mathbb{S}_4)) \cap C^1((0,1], Q^{s-1}(\mathbb{R}^3,\mathbb{S}_4)),$$
$$u_{\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], Q^s(\mathbb{R}^3)) \cap C^1((0,1], Q^{s-1}((\mathbb{R}^3)),$$
$$u_{\gamma,\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], Q^s(\mathbb{R}^3)) \cap C^1((0,1], Q^{s-1}((\mathbb{R}^3)),$$
$$\delta\zeta_{\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], Q^s(\mathbb{R}^3)) \cap C^1((0,1], Q^{s-1}(\mathbb{R}^3)),$$
$$z_{i,\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], Q^s(\mathbb{R}^3),\mathbb{R}^3)) \cap C^1((0,1], Q^{s-1}(\mathbb{R}^3,\mathbb{R}^3)),$$

for $\epsilon \in (0,\epsilon_0)$, and

$$\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], Q^{s+2}(\mathbb{R}^3)) \cap C^1((0,1], Q^{s+1}(\mathbb{R}^3)),$$
$$\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}} \in C^0((0,1], H^s(\mathbb{R}^3)) \cap C^1((0,1], H^{s-1}(\mathbb{R}^3)),$$

such that

(i) $\{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}(t,x), u_{\epsilon,\vec{\mathbf{y}}}(t,x), \delta\zeta_{\epsilon,\vec{\mathbf{y}}}(t,x), z_{i,\epsilon,\vec{\mathbf{y}}}(t,x)\}$ determines, a 1-parameter family of solutions to the Einstein-Euler equations in the wave gauge on M_R,

(ii) $\{\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}(t,x), \mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}(t,x) := \delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}} + \mathring{\zeta}_H, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}(t,x) := \mathring{E}(t)^{-2}\delta^{ij}\mathring{z}_{j,\epsilon,\vec{\mathbf{y}}}(t,x)\}$, with $\mathring{\zeta}_H$ and $\mathring{E}$, respectively, solves the conformal cosmological Poisson-Euler equations on M_N with initial data

$$\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}|_{t=1} = \ln\left(\frac{4C_0\Lambda}{(C_0-1)^2} + \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\right) \quad \text{and} \quad \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}|_{t=1} = \breve{z}^i_{\epsilon,\vec{\mathbf{y}}}, \tag{29}$$

(iii) the uniform bounds

$$\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],H^s)} + \|\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],H^{s+2})} + \|\mathring{z}_{j,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0]\times H^s)} + \|\delta\zeta_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^s)} \\ + \|z_{j,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^s)} \lesssim 1$$

and

$$\|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0],Q^s)} + \|u^{\mu\nu}_{\gamma,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^s)} + \|u_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^s)} + \|u_{\gamma,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^s)} \lesssim 1,$$

hold for $\epsilon \in (0,\epsilon_0)$,

(iv) and the uniform error estimates

$$\|\delta\zeta_{\epsilon,\vec{\mathbf{y}}} - \delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^{s-1})} + \|z_{j,\epsilon,\vec{\mathbf{y}}} - \mathring{z}_{j,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0]\times Q^{s-1})} \lesssim \epsilon,$$

$$\|u^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0],Q^{s-1})} + \|u^{\mu\nu}_{k,\epsilon,\vec{\mathbf{y}}} - \delta^\mu_0\delta^\nu_0\partial_k\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^{s-1})} + \|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^{s-1})} \lesssim \epsilon$$

and

$$\|u_{\gamma,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^{s-1})} + \|u_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],Q^{s-1})} \lesssim \epsilon$$

hold for $\epsilon \in (0,\epsilon_0)$.