Nonlinear singularity-free cosmological solutions and scalarization in ESGB theories

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Motivation

- General Relativity predicts singularities (e.g., Big Bang and black hole singularities).
- Singularities mark the breakdown of classical GR.
- Modified gravity theories (e.g., including EsGB and the string-inspired gravity EdGB) aim to resolve these.
- **Core Challenge**: Construct global singularity-free cosmological models in modified gravity.
- **Achievement:** First rigorous proof of globally singularity-free solutions, aligning with numerical results.

Nonlinear singularity-free cosmological solutions and scalarization in EsGB gravity

(joint work with Chihang He and Jinhua Wang)

Introduction on EsGB theory

Core Challenge: Constructing singularity-free cosmological models in Einstein-Scalar-Gauss-Bonnet (EsGB) that avoid Big Bang singularity
EsGB Action:

$$S_{\mathrm{ESGB}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\underbrace{R}_{\mathrm{GR}}\underbrace{-2\partial_\mu\phi\partial^\mu\phi - V_\phi}_{\text{scalar field}}\underbrace{-\lambda f(\phi)R^2_{\mathrm{GB}}}_{\text{GB term}}\right),$$

where $R^2_{\mathrm{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \leftarrow$ Gauss-Bonnet term
FLRW Metric (Homog. and Isotro.):

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

Physical Context: EsGB gravity with quadratic coupling

$$V_\phi = 0, \quad \lambda = 1 \quad \text{and} \quad f(\phi) = \frac{1}{2}\phi^2$$

Mathematical Framework

Field Equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Gamma_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - (\nabla\phi)^2 g_{\mu\nu}, \leftarrow \boxed{\text{Einstein Eq.}}$$

and

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\partial^\mu\phi\right] - \frac{1}{4}\phi R^2_{\mathrm{GB}} = 0, \leftarrow \boxed{\text{Scalar Field Eq.}}$$

where $\Gamma_{\mu\nu}$ is defined as

$$\Gamma_{\mu\nu} = 2R\nabla_\mu\nabla_\nu f + 4(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\nabla^\alpha\nabla_\alpha f - 8R_{\alpha(\mu}\nabla^\alpha\nabla_{\nu)} f + 4R^{\alpha\beta}g_{\mu\nu}\nabla_\alpha\nabla_\beta f - 4R^\beta{}_{\mu\alpha\nu}\nabla^\alpha\nabla_\beta f.$$

Field Equations for FLRW

Under Assumptions: Homog. and Isotro.

$$3H^2 = \dot{\phi}^2 + 12H^3\phi\dot{\phi}, \leftarrow \boxed{\text{Friedmann equation; Hamiltonian constraint}}$$

$$2\dot{H} + 3H^2 = -\dot{\phi}^2 + 8(\dot{H} + H^2)H\phi\dot{\phi} + 4H^2\left(\dot{\phi}^2 + \phi\ddot{\phi}\right) \leftarrow \boxed{\text{Einstein Eq.}}$$

$$\ddot{\phi} + 3\dot{\phi}H \qquad \underbrace{+6\phi\,\overbrace{(H^2 + \dot{H})H^2}^{<0 \text{ for } t>0}}_{\text{Tachyonic instability} \implies \text{scalarization}} \qquad = 0, \leftarrow \boxed{\text{Scalar Field Eq.}}$$

Main goal: The analysis of this nonlinear ODE system

Main Theorem 1.1: Global Existence and Bounds

Initial Conditions: $(a_0, \beta, \alpha) := (a, H, \phi)|_{t=0}$ satisfies

$$a_0 > 0, \quad \beta \in (0, \sqrt{3}/3), \quad \alpha = 0$$

and

$$\dot{\phi}(0) > 0 \leftarrow \boxed{\text{break } \mathbb{Z}_2 \text{ symmetry } \phi \to -\phi}$$

Key Results:

- Exist a unique globally singularity-free FLRW solution $(g, \phi) \in C^2((-\infty, +\infty))$ (Note $\forall t \in (-\infty, +\infty)$)
- Hubble parameter $H(t) > 0$ with $\lim_{t\to+\infty} H(t) = 0$ and $\lim_{t\to-\infty} H(t) =$ some positive constant
- Scalar field $\phi(t)$ evolves monotonically
- Explicit bounds for $H(t)$ and $\phi(t)$ provided for $t < 0$ and $t > 0$ (see next pages).

Explicit Bounds for $t < 0$:

$$\mathfrak{L}(t) < H(t) < \frac{2\beta + \sqrt{2} - \left(\sqrt{2} - 2\beta\right) e^{3\sqrt{2}t}}{\sqrt{2}\left(2\beta + \sqrt{2} + \left(\sqrt{2} - 2\beta\right) e^{3\sqrt{2}t}\right)},$$

and

$$\frac{\sqrt{3}}{12}(1 - e^{-12t}) < \phi(t) < \mathfrak{H}(t),$$

where $\mathfrak{L}(t)$ and $\mathfrak{H}(t)$ are defined by

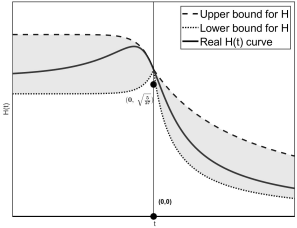
$$\mathfrak{L}(t) := \begin{cases} \frac{227\beta}{2e^{\frac{454\beta t}{45}} + 225} > \beta, & \text{if } 0 < \beta \leq \sqrt{\frac{5}{27}}, \\ \frac{2\beta}{5\beta - (5\beta - 2)e^{4t}} > \frac{2}{5}, & \text{if } \sqrt{\frac{5}{27}} < \beta < \frac{\sqrt{3}}{3}. \end{cases}$$

and

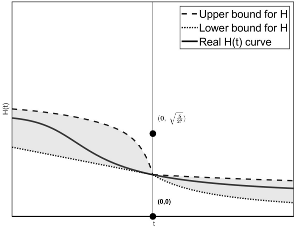
$$\mathfrak{H}(t) := \begin{cases} \frac{\sqrt{3}}{6\beta^2}(1 - e^{-6\beta^3 t}), & \text{if } 0 < \beta \leq \sqrt{\frac{5}{27}}, \\ \frac{25\sqrt{3}}{24}(1 - e^{-\frac{48}{125}t}), & \text{if } \sqrt{\frac{5}{27}} < \beta < \frac{\sqrt{3}}{3}. \end{cases}$$

(See the article for explicit bounds of a and $\dot{\phi}$, omitted here.)

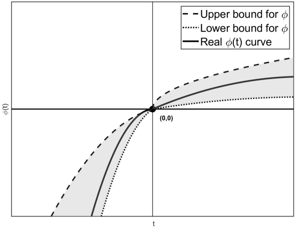
Fig. on Bounds for H and ϕ



(a) Bounds for H ($\beta > \sqrt{5/27}$)



(b) Bounds for H ($\beta \leq \sqrt{5/27}$)



(c) Bounds for ϕ

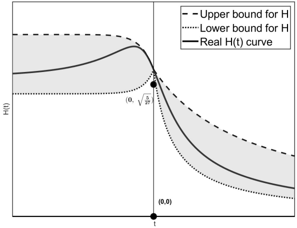
Explicit Bounds for $t > 0$:

$$\frac{1}{5t + 1/\beta} < H(t) < \frac{1}{t + 1/\beta},$$

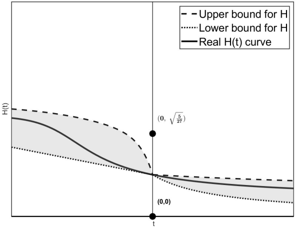
and

$$\frac{-1 + \sqrt{1 + \frac{24\beta^2}{5}\ln(1 + 5\beta t)}}{4\sqrt{3}\beta^2} < \phi(t) < \sqrt{3}\ln(\beta t + 1).$$

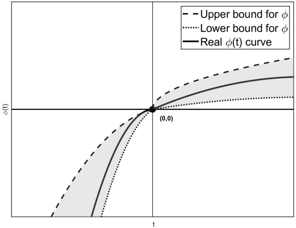
Fig. on Bounds for H and ϕ



(a) Bounds for H ($\beta > \sqrt{5/27}$)



(b) Bounds for H ($\beta \leq \sqrt{5/27}$)



(c) Bounds for ϕ

Theorem 1.2: Scalarization via Tachyonic Instability

Admissible Data Set:

$$\underbrace{\mathcal{A}}_{\text{Nonempty}} := \left\{ (a_0, \beta, \alpha) \in (0, +\infty) \times (0, \sqrt{3}/3) \times [0, +\infty) \mid -5\beta^2 < \kappa(\alpha, \beta) < -\beta^2 \right\}$$

and $\dot{\phi}(0) > 0$.

Then exist a unique global, FLRW solution $(g, \phi) \in C^2([0, +\infty))$

Explicit Bounds for $t > 0$:

$$\frac{1}{5t + 1/\beta} < H(t) < \frac{1}{t + 1/\beta} \quad \text{and} \quad \frac{-1 + \sqrt{1 + \frac{48\beta^2}{5} \ln(1 + 5\beta t)}}{4\sqrt{3}\beta^2} < \phi(t) < \sqrt{3} \ln(\beta t + 1) + \alpha$$

Scalarization (ϕ increasing) Mechanism

Triggered by tachyonic instability (Gauss-Bonnet term-induced), scalarization refers to:

Trivial scalar field ($\phi = 0$) $\to$ Nontrivial scalar "hair"

(Analogous to Jeans instability — same linear origin, distinct nonlinear evolution)

Scalarization via Tachyonic Instability

Scalarization Phenomenon:

- Nontrivial scalar field growth due to curvature-induced instability (tachyonic instability) from Gauss–Bonnet term. A toy model for tachyonic instabilities.
- Fully nonlinear regime beyond linear perturbation theory. A nonlinear spontaneous scalarization

$$\boxed{\text{tachyonic instability} \overset{\text{on the linear level}}{=} \text{Jeans instability}}$$

Model:

$$\partial_t^2 \phi - \Delta\phi - m^2\phi = 0 \quad \overset{\text{Fourier on } x}{\Longrightarrow} \quad \partial_t^2\hat{\phi} + k^2\hat{\phi} - m^2\hat{\phi} = 0$$

If long wavelength $k \to 0 \implies$ an exponential growth mode $e^{\sqrt{m^2-k^2}\,t}$.

- However, their nonlinear evolutions are governed by the underlying physical models, leading to distinct behaviors due to the different nonlinear terms.
- Nonlinear Jeans instabilities are studied by my own series works (2022-2025).

Recall

Under Assumptions: Homog. and Isotro.

$$3H^2 = \dot{\phi}^2 + 12H^3\phi\dot{\phi}, \leftarrow \boxed{\text{Friedmann equation; Hamiltonian constraint}}$$

$$2\dot{H} + 3H^2 = -\dot{\phi}^2 + 8(\dot{H} + H^2)H\phi\dot{\phi} + 4H^2\left(\dot{\phi}^2 + \phi\ddot{\phi}\right) \leftarrow \boxed{\text{Einstein Eq.}}$$

$$\ddot{\phi} + 3\dot{\phi}H \quad \underbrace{+6\phi\overbrace{(H^2 + \dot{H})H^2}^{<0 \text{ for } t>0}}_{\text{Tachyonic instability} \Longrightarrow \text{ scalarization}} \quad = 0, \leftarrow \boxed{\text{Scalar Field Eq.}}$$

Main goal: The analysis of this nonlinear ODE system

Table: Historical Development of Singularity-Free Cosmology and Scalarization

Time Period	Phase	Key Developments	Researchers
1980-1993	Foundations, Early Exploration in GR	• Theoretical proposals • Wave function approaches • Early singularity avoidance	Starobinsky (1980) Hartle & Hawking (1983) Brandenberger, Mukhanov, Trodden & Sornborger (1993)
1994-1999	ESGB Breakthroughs	• First singularity-free solutions • Quadratic coupling (k=0) • Arbitrary curvature extension	Antoniadis, Rizos & Tamvakis. (1994) Rizos & Tamvakis (1994) Easther & Maeda (1996) Kanti, Rizos & Tamvakis. (1999)
1998-2017	Analytical Refinements	• Instability analyses • Early-time approximations • Gauss-Bonnet inflation	Kawai, Sakagami & Soda (1998) Kanti, Gannouji & Dadhich (2015) Sberna & Pani (2017)
2017	Numerical Study	• Phase space analysis • Scalar field evolution • Hubble parameter study	Sberna (2017)
1993-Present	Scalarization Research	• DEF model development (Tachyonic instabilities) • Compact objects scalarization • Cosmological extensions	Damour & Esposito-Farèse (1993, 1996) Various groups......

Proofs of the Main Theorems

Steps:

1. establishing local existence (ODE system on $\mathcal{U} := (a, H, \phi, \dot{\phi})$ and local existence theorem of ODE for $\frac{d}{dt}\mathcal{U} = \mathcal{F}(\mathcal{U})$)
2. (Focus on this! Core technical ingredient) deriving bounds via

$$\begin{cases} \text{Power Identity} \\ \text{First-Hit argument} \end{cases} \Longrightarrow \underbrace{\text{decoupled}}_{\text{Fail in exponential coupling!}} \quad \text{diff. ineq's for } H$$

on $t \in (\mathcal{T}_-, \mathcal{T}_+)$ which is the max. interval of existence of sol. Then bounds for ϕ by the constraint equation $3H^2 = \dot{\phi}^2 + 12H^3\phi\dot{\phi}$,

$$\dot{\phi}(t) = -6\phi(t)H^3(t) + \sqrt{(6\phi(t)H^3(t))^2 + 3H^2(t)} > 0$$

3. extending the solution globally (by continuation arguments)

Recall

Under Assumptions: Homog. and Isotro.

$$3H^2 = \dot{\phi}^2 + 12H^3\phi\dot{\phi}, \leftarrow \boxed{\text{Friedmann equation; Hamiltonian constraint}}$$

$$2\dot{H} + 3H^2 = -\dot{\phi}^2 + 8(\dot{H} + H^2)H\phi\dot{\phi} + 4H^2\left(\dot{\phi}^2 + \phi\ddot{\phi}\right) \leftarrow \boxed{\text{Einstein Eq.}}$$

$$\ddot{\phi} + 3\dot{\phi}H \qquad \underbrace{+6\phi\,\overbrace{(H^2 + \dot{H})H^2}^{<0 \text{ for } t>0}}_{\text{Tachyonic instability} \implies \text{ scalarization}} \qquad = 0, \leftarrow \boxed{\text{Scalar Field Eq.}}$$

Main goal: The analysis of this nonlinear ODE system

Power Identity & First-Hit Argument (Core Tools)

Power Identity (derived from the derivatives of the constraint and scalar field equations)$\leftarrow$ Acts as a "litmus test"

$$\mathscr{P} := H\left(\left(2H^2 - 1 - \frac{\dot{H}}{3H^2}\right)\dot{\phi}^2 - 8H^3\phi\dot{\phi} - 12H^6\left(1 + \frac{\dot{H}}{H^2}\right)\phi^2\right) = 0 \leftarrow \boxed{\text{How to derive?}}$$

Key Properties:

- Algebraic constraint satisfied by solutions at all times
- Serves as "energy conservation law" for the system
- Enables decoupling of Hubble parameter H from scalar field ϕ

First-Hit Argument Strategy:

1. Define auxiliary quantities $B_\ell(t)$ (e.g., $B_1 = \dot{H} + 5H^2$) that capture the dominant behaviors in the evolution of H.
2. Prove sign preservation ($B_\ell(t) > 0$ or < 0 for all $t > 0$ or < 0) via contradiction with power identity$\leftarrow$ How to prove?
3. Derive differential inequalities for H (e.g., $B_1 > 0 \implies \dot{H} > -5H^2$ for all $t > 0$)
4. Obtain explicit bounds through comparison theorems ($H(t) > \frac{1}{5t+\frac{1}{\beta}}$ for all $t > 0$)

Derivations of Power Identity

Recall:

$$3H^2 = \dot{\phi}^2 + 12H^3\phi\dot{\phi}, \leftarrow \boxed{\text{Hamiltonian constraint}}$$

$$2\dot{H} + 3H^2 = -\dot{\phi}^2 + 8(\dot{H} + H^2)H\phi\dot{\phi} + 4H^2\left(\dot{\phi}^2 + \phi\ddot{\phi}\right) \leftarrow \boxed{\text{Einstein Eq.}}$$

$$\ddot{\phi} + 3\dot{\phi}H \quad \underbrace{+6\phi\overbrace{(H^2 + \dot{H})H^2}^{<0 \text{ for } t>0}}_{\text{Tachyonic instability} \implies \text{ scalarization}} \quad = 0, \leftarrow \boxed{\text{Scalar Field Eq.}}$$

Then

$$\left\{\begin{matrix} \text{Einstein Eq.} \\ \text{Scalar Field Eq.} \end{matrix}\right\} \implies \underbrace{\partial_t(3H^2 - \dot{\phi}^2 - 12H^3\phi\dot{\phi})}_{\ddot{\phi} \text{ appears and is eliminated using the scalar field eq.}} = 0$$

$\implies$ Power Identity

Hierarchical Estimation Strategy

Core Approach: Decouple H from ϕ via differential inequalities

Auxiliary Quantities:

$$B_1 = \dot{H} + 5H^2 > 0 \quad \text{(Future lower bound)} \leftarrow \text{As an example}$$

$$B_2 = \dot{H} - 6H^4 - 6H^2 < 0 \quad \text{(Past lower bound} \to 0\text{, as } t \to -\infty)$$

$$B_3 = \dot{H} + H^2 < 0 \quad \text{(Future upper bound)}$$

$$B_4 = \dot{H} - 3H^2 + \frac{3}{2} > 0 \quad \text{(Past upper bound)}$$

$$B_5 = \begin{cases} \dot{H}(t) - 10H^2(t) + \frac{454\beta}{45} H(t), & \text{if } 0 < \beta \leq \sqrt{\frac{5}{27}}, \\ \dot{H}(t) - 10H^2(t) + 4H(t), & \text{if } \sqrt{\frac{5}{27}} < \beta < \frac{\sqrt{3}}{3} \end{cases}$$

(Improved past lower bounds, the limit tends to a constant as $t \to -\infty$)

1. Prove sign preservation via power identity contradictions
2. Solve resulting Riccati-type inequalities
3. Obtain explicit time-dependent bounds
4. Extend to scalar field via hierarchical estimation

A Simple Example: Brief Ideas on Lower Bound for $H(t)$ on $(0, \mathscr{T}_+)$

Target

Prove that on the interval $t \in (0, \mathscr{T}_+)$, the Hubble parameter satisfies:

$$H(t) > \frac{1}{5t + \frac{1}{\beta}}$$

- This is the simplest case, which makes it easy to illustrate the basic ideas. The other bounds are more technical; please refer to the article for details.

Step 1: Construct Auxiliary Function $B_1(t)$

Define the auxiliary function:

$$\boxed{B_1(t) := \dot{H}(t) + 5H^2(t)}$$

Rationale: This specific form helps establish a lower bound for $H(t)$ through a Riccati-type differential inequality.

Step 2: Establish Differential Inequality and Solve

Initial Condition Verification: Under the initial data of the Main Theorems, we can check that $B_1(0) > 0$ holds.

Differential Inequality: From $B_1(t) > 0$ for the max. interval of existence, we obtain:

$$\dot{H} + 5H^2 > 0 \quad \Rightarrow \quad \dot{H} > -5H^2 \leftarrow \boxed{\text{Riccati Eq.}}$$

Comparison Function: Define $\underline{H}_+(t)$ satisfying:

$$\dot{\underline{H}}_+ = -5\underline{H}_+^2, \quad \underline{H}_+(0) = \beta$$

By comparison principle: $H(t) > \underline{H}_+(t)$

Separation of Variables:

$$\frac{d\underline{H}_+}{dt} = -5\underline{H}_+^2 \quad \Rightarrow \quad \frac{d\underline{H}_+}{\underline{H}_+^2} = -5dt$$

Integrate both sides:

$$\underline{H}_+(t) = \frac{1}{5t + \frac{1}{\beta}} \implies \boxed{H(t) > \underline{H}_+(t) > \frac{1}{5t + \frac{1}{\beta}} > 0}$$

Step 3: Prove $B_1(t) > 0$ on $(0, \mathcal{T}_+)$

Initial Condition Verification (Recall): Under the initial data of the Main Theorems, we can check that $B_1(0) > 0$ holds.

First-Hit Argument:

- By continuity, there exists a maximal interval $(0, T_{\max})$ where $B_1(t) > 0$
- Then by continuity: $B_1(T_{\max}) = 0$
- Assume for contradiction that $T_{\max} < \mathcal{T}_+$

Deriving the Contradiction by Power Identity Application: At $t = T_{\max}$, using the power identity and $B_1 = 0$ (i.e., $\dot{H} = -5H^2$):

$$\mathscr{P}(T_{\max}) := H\left(\left(2H^2 - 1 - \frac{\dot{H}}{3H^2}\right)\dot{\phi}^2 - 8H^3\phi\dot{\phi} - 12H^6\left(1 + \frac{\dot{H}}{H^2}\right)\phi^2\right)\Big|_{T_{\max}} = 0$$

$$\Downarrow \dot{H} = -5H^2$$

$$\mathscr{P}(T_{\max}) = H\left(\left(2H^2 + \frac{2}{3}\right)\dot{\phi}^2 - 8H^3\phi\dot{\phi} + 48H^6\phi^2\right)$$

$$\Downarrow$$

$$\mathscr{P}(T_{\max}) = H\left[2H^2\dot{\phi}^2 + 2\left(\frac{\dot{\phi}}{\sqrt{3}} - 2\sqrt{3}H^3\phi\right)^2 + 24H^6\phi^2\right]$$

Step 3 (cont.): Conclusion

Rewriting in perfect square form:

$$\mathscr{P}(T_{\max}) = H\left[\underbrace{2H^2\dot{\phi}^2}_{>0} + \underbrace{2\left(\frac{\dot{\phi}}{\sqrt{3}} - 2\sqrt{3}H^3\phi\right)^2 + 24H^6\phi^2}_{\geq 0}\right] > 0$$

- $H(T_{\max}) > 0$ (from continuity and $H(t) > \frac{1}{5t+\frac{1}{\beta}} > 0$)
- $\dot{\phi}(T_{\max}) \neq 0$ since

$$\dot{\phi}(T) = -6\phi(T)H^3(T) + \sqrt{(6\phi(T)H^3(T))^2 + \underbrace{3H^2(T)}_{>0}}$$
$$> 6|\phi(T)H^3(T)| - 6\phi(T)H^3(T) \geq 0,$$

- All terms are non-negative, with the first term strictly positive

Contradiction: $\mathscr{P}(T_{\max}) > 0$ but power identity requires $\mathscr{P} \equiv 0$.

Sign Permanence: The contradiction implies our assumption was false, hence:

$$T_{\max} = \mathscr{T}_+ \quad \Rightarrow \quad B_1(t) > 0 \ \forall t \in (0, \mathscr{T}_+)$$

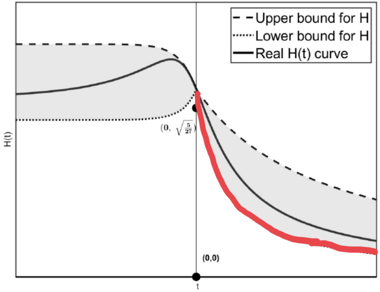
Step 4: Final Result

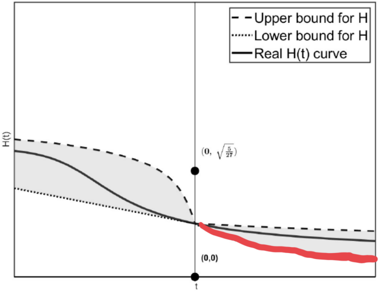
Lower Bound Conclusion:

$$H(t) > \underline{H}_{+}(t) = \frac{1}{5t + \frac{1}{\beta}}, \quad \forall t \in (0, \mathcal{T}_{+})$$

Achievement

We have successfully established the desired lower bound for the Hubble parameter on the forward time interval using $B_1(t)$.





Difficulties

- This is the simplest case, which makes it easy to illustrate the basic ideas. The other bounds are more technical; please refer to the article for details.
- The main difficulty lies in how to define B_ℓ in a way that preserves the required sign, is easy to solve, and yields higher accuracy.

Summary

- **Key Contributions:**
  - First rigorous proof of globally singularity-free solutions
  - Mathematical proof of nonlinear spontaneous scalarization
  - Novel framework: Power identity + First-hit argument $\Longrightarrow$ decoupled differential inequalities (generalizable to broad nonlinear ODE problems).
- **Significance:** Provides mathematical foundation for numerical results and explores tachyonic instability mechanisms

Nonlinear singularity-free cosmological solutions in EdGB gravity

(joint work with Chihang He)

Motivation

- General Relativity predicts singularities (e.g., Big Bang).
- **Goal:** Construct cosmological models that avoid the Big Bang singularity.
- Modified gravity theories aim to resolve these.
- **Theory:** Einstein-dilaton-Gauss-Bonnet (EdGB) gravity, inspired by superstring theory.
- EdGB (Einstein–Dilaton–Gauss–Bonnet) gravity is a **string-inspired** theory from low-energy superstring effective action, addressing:
  - Cosmic acceleration (dark energy problem)
  - Cosmic inflation
  - Includes dilaton scalar field ϕ and Gauss–Bonnet term.
  - Able to avoid singularities.
- We aim to prove **global singularity-free solutions** in EdGB gravity with exponential coupling.
- **Challenge:** Strong nonlinearities from exponential coupling $f(\phi) = e^{\phi}$ (can not decouple H).

Introduction on EdGB theory

Action:

$$S_{\text{EsGB}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_\phi - \lambda\frac{f(\phi)}{8}R^2_{\text{GB}}\right)$$

FLRW Metric:

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

Physical Context: EdGB gravity with exponential coupling (can not reduce to General Relativity)

$$V_\phi = 0, \quad \lambda = 1 \quad \text{and} \quad f(\phi) = e^\phi.$$

Field Equations:

$$3H^2 - 3e^\phi\dot{\phi}H^3 = \frac{\dot{\phi}^2}{2} \leftarrow \boxed{\text{Hamiltonian constraint}}$$

$$2\dot{H} + 3H^2 = -\frac{\dot{\phi}^2}{2} + 2e^\phi\dot{\phi}H(H^2 + \dot{H}) + e^\phi H^2(\dot{\phi}^2 + \phi\ddot{\phi}) \leftarrow \boxed{\text{Einstein Eq.}}$$

$$\ddot{\phi} = -3H\dot{\phi} - 3e^\phi H^2(H^2 + \dot{H}) \leftarrow \boxed{\text{Scalar Field Eq.}}$$

Main Theorem and Results

Under initial conditions $(a_0, \beta, \alpha) := (a, H, \phi)|_{t=0}$ where $a_0 > 0$, $\alpha = 0$, $\beta \in (0, \sqrt{6}/6)$, $\dot{\phi}(0) < 0$:

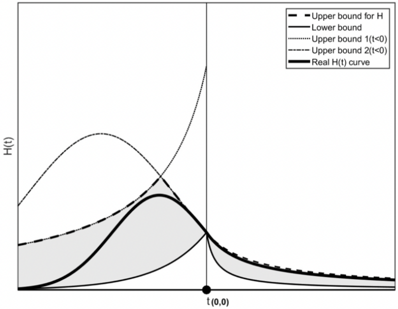
- Exist a unique globally singularity-free FLRW solution $(g, \phi) \in C^2((-\infty, +\infty))$ (Note $\forall t \in (-\infty, +\infty)$).
- $H(t) > 0$ and $\lim_{t \to \pm\infty} H(t) = 0$.
- $\phi(t)$ evolves monotonically.
- Explicit bounds for $H(t)$ and $\phi(t)$ provided for $t < 0$ and $t > 0$ (see next pages).

(1) For $t \in (0, +\infty)$, H satisfies

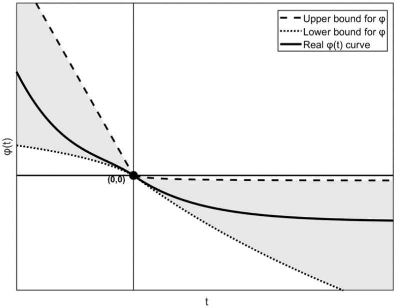
$$\frac{1}{5t + \frac{1}{\beta}} < H(t) < \frac{1}{\frac{1}{2}t + \frac{1}{\beta}}$$

and ϕ satisfies

$$-(12\beta^2 + 2\sqrt{6}) \ln\left(\frac{1}{2}\beta t + 1\right) < \phi(t) < -\ln\left(1 + \frac{3}{5}\left(\beta^2 - \frac{1}{(5t + \frac{1}{\beta})^2}\right)\right).$$



(A) Bounds for H



(B) Bounds for ϕ

(2) For $t \in (-\infty, 0)$, H satisfies

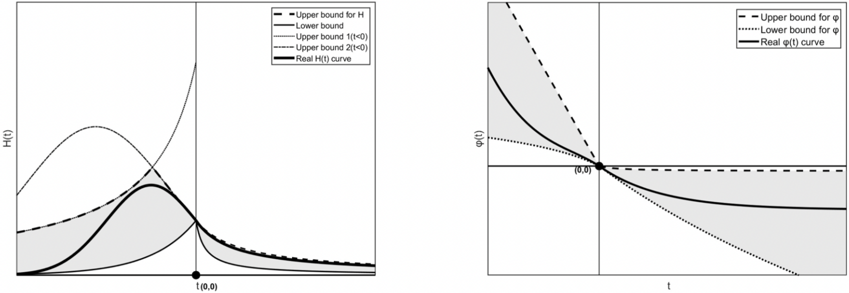
$$H(t) < \min\left\{ \left(e^{6\gamma t}\left(\frac{1}{\beta^2} - \frac{4\beta}{5\gamma} + \frac{2\theta}{15\gamma^2}\right) - \frac{2(-6\beta\gamma + \theta + 6\gamma\theta t)}{15\gamma^2} \right)^{-\frac{1}{2}}, \ \frac{\gamma}{-3\beta^3 t + 1} \right\},$$

$$H(t) > \left[\beta^{-1/4} - \frac{\beta^{-1/4}}{4(m+1)} \left(1 - (-3\beta^3 t + 1)^{m+1} \right) \right]^{-4} > 0,$$

and ϕ satisfies

$$\ln(-3\beta^3 t + 1)^2 < \phi < \frac{\sqrt{6}\gamma}{3\beta^3} \ln(-3\beta^3 t + 1) + \frac{2\beta^3}{4\gamma + \beta^3} \left[(-3\beta^3 t + 1)^{\frac{4\gamma+\beta^3}{\beta^3}} - 1 \right],$$

where $\theta := \frac{2\left(9\beta^6 + \sqrt{9\beta^6+12}\,\beta^3\right)}{\left(\sqrt{9\beta^6+12}+3\beta^3\right)^2}$, $\quad \gamma = \frac{3\beta^3+\sqrt{9\beta^6+12}}{2}$ and $m = \frac{107\sqrt{9\beta^6+12}+309\beta^3}{60\beta^3}$.



Proofs of the Main Theorems

Steps:

1. establishing local existence (ODE system on $\mathcal{U} := (a, H, \phi, \dot{\phi})$ and local existence theorem of ODE for $\frac{d}{dt}\mathcal{U} = \mathcal{F}(\mathcal{U})$)
2. (Focus on this! Core technical ingredient) deriving bounds via

   $$\begin{cases} \text{Power Identity} \\ \text{First-Hit argument} \end{cases} \implies \text{non-decoupled diff. ineq's for } H \text{ and } \phi$$

   on $t \in (\mathcal{T}_-, \mathcal{T}_+)$ which is the max. interval of existence of sol. Then bounds for ϕ by the constraint equation $3H^2 - 3e^\phi \dot{\phi} H^3 = \frac{\dot{\phi}^2}{2}$,

   $$\dot{\phi} = -3H^3 e^\phi - \sqrt{(3H^3 e^\phi)^2 + 6H^2} < 0 \leftarrow \text{focus on negative branch}$$

3. extending the solution globally (by continuation arguments)

Power Identity & First-Hit Argument (Core Tools)

Power Identity (derived from the derivatives of the constraint and scalar field equations)$\leftarrow$ Acts as a "litmus test"

$$\mathscr{P} = H\left(\left(1 - H^2 e^{\phi} + \frac{\dot{H}}{3H^2}\right)\dot{\phi}^2 + 4H^3\dot{\phi}e^{\phi} + 3H^6 e^{2\phi}\left(1 + \frac{\dot{H}}{H^2}\right)\right) = 0 \leftarrow \boxed{\text{similar to prev.}}$$

Key Properties:

- Algebraic constraint satisfied by solutions at all times
- Serves as "energy conservation law" for the system
- (New Challenges) Possible failure of decoupling H from scalar field ϕ

First-Hit Argument Strategy (Similar to the previous case):

1. Define auxiliary quantities $D_\ell(t)$ (e.g., $D_1 = \dot{H} + 5H^2$).
2. Show $D_\ell(0)$ preserves a specific sign (e.g., $D_1(0) > 0$).
3. Assume a "first time" $T_{\max}$ where $D_\ell(T_{\max}) = 0$.
4. Substitute into $\mathscr{P} = 0 \Rightarrow 0 < \mathscr{P} = 0 \Rightarrow D_\ell(t)$ preserves sign $\forall t$.
5. Derive differential inequalities for $H \leftarrow$??? $\boxed{\text{Possible failure!!!}}$
6. Obtain explicit bounds through comparison theorems

Analysis for Future Evolution ($t > 0$)

- The proof for the future is relatively direct, which is similar to the analysis in the previous case–We can derive decoupled differential inequalities for H.

Key Quantities:

$$D_1 = \dot{H} + 5H^2 > 0 \quad \Rightarrow \quad H(t) > \frac{1}{5t + 1/\beta}$$

$$D_2 = \dot{H} + \frac{1}{2}H^2 < 0 \quad \Rightarrow \quad H(t) < \frac{1}{\frac{1}{2}t + 1/\beta}$$

Results:

- Hubble parameter $H(t)$ remains positive and vanishes as $t \to +\infty$.
- Scalar field $\phi(t)$ decreases monotonically by the constraint $\dot{\phi} = -3H^3e^{\phi} - \sqrt{(3H^3e^{\phi})^2 + 6H^2} < 0$.
- No future singularity.

Analysis for Past Evolution ($t < 0$)

Challenges:

- Exponential coupling e^{ϕ} becomes unbounded as $\phi \to +\infty$ ($t \to -\infty$).
- Cannot decouple H and ϕ (unlike quadratic coupling case).

Key Quantities:

$$D_3 = \dot{H} - 3H^4 e^{\phi} + H^2 < 0$$
$$D_4 = \dot{H} - \frac{12}{5} H^4 e^{\phi} + 3H^2 > 0$$
$$D_5 = \dot{H} - H^2 + 3e^{-\phi} > 0$$

- The power identity and the "first hit argument" is again used to prove that these preserve their specific signs ($D_3 < 0$, $D_4 > 0$, $D_5 > 0$) for all past time $t \in (\mathcal{T}_-, 0)$.
- However, because of the coupling and the *unbounded* nature of e^{ϕ}, these inequalities $D_3 < 0$, $D_4 > 0$, and $D_5 > 0$ are difficult to solve directly.

Hierarchical Estimation Strategy ($t < 0$)

Key Problem

How can we simultaneously (since can not decouple) solve $D_3 < 0$, $D_4 > 0$, and $D_5 > 0$ to determine the upper and lower bounds of H and ϕ?

Ideas

- examine the behavior of the composite quantities $H^3 e^\phi$, He^ϕ, and $H^{11/4} e^\phi$
- Instead of attempting to control H through the unbounded exponential e^ϕ, our approach is to control H using the better-behaved quantities He^ϕ and $H^{11/4} e^\phi$, while controlling e^ϕ through $H^3 e^\phi$.
- introduce differential inequalities for He^ϕ, $H^{11/4} e^\phi$, and $H^3 e^\phi$ $\leftarrow$ How???

This yields a coupled system of differential inequalities linking H, ϕ, He^ϕ, $H^{11/4} e^\phi$, and $H^3 e^\phi$

Differential inequalities for He^ϕ, $H^{11/4}e^\phi$, and H^3e^ϕ

Direct computations:

$$\begin{aligned}
\frac{d(H^3e^\phi)}{dt} =& 3H^2\dot{H}e^\phi + H^3e^\phi\dot{\phi},\\
\frac{d(He^\phi)}{dt} =& \dot{H}e^\phi + He^\phi\dot{\phi},\\
\frac{d(H^{\frac{11}{4}}e^\phi)}{dt} =& \frac{11}{4}H^{\frac{7}{4}}\dot{H}e^\phi + H^{\frac{11}{4}}e^\phi\dot{\phi}.
\end{aligned}$$

- (Replacing $\dot{H}$) Substituting $D_3 < 0$, $D_4 > 0$, and $D_5 > 0$ ($\implies$ $\dot{H} <$ or $>$...)
- (Replacing $\dot{\phi}$) $\dot{\phi} < -6H^3e^\phi$ and $\dot{\phi} > -6H^3e^\phi - \sqrt{6}H$ (from the constraint $\dot{\phi}(t) = -3H^3(t)e^{\phi(t)} - \sqrt{\left(3H^3(t)e^{\phi(t)}\right)^2 + 6H^2(t)} < 0$)

Conclusion ($\star$):

$$\begin{aligned}
\frac{d(H^3e^\phi)}{dt} <& 3H^6e^{2\phi} - 3H^4e^\phi \quad \text{and} \quad \frac{d(H^3e^\phi)}{dt} > \frac{6}{5}H^6e^{2\phi} - 12H^4e^\phi,\\
\frac{d(He^\phi)}{dt} <& -3H^4e^{2\phi} - H^2e^\phi \quad \text{and} \quad \frac{d(H^{\frac{11}{4}}e^\phi)}{dt} > \frac{1}{10}H^{\frac{11}{4}}e^\phi\left(6H^3e^\phi - 107H\right)
\end{aligned}$$

Closed System via Variable Transformation

Variable Transformation:

$$y = H, \quad w = e^{\phi}, \quad z = H^3 e^{\phi}, \quad v = He^{\phi}, \quad p = H^{11/4} e^{\phi}$$

Using the sign conditions $D_3 < 0,\ D_4 > 0,\ D_5 > 0$, **together with Conclusion** $(\star)$ **and the bounds**

$$\dot{\phi} < -6H^3 e^{\phi}, \qquad \dot{\phi} > -6H^3 e^{\phi} - \sqrt{6}\, H,$$

the variables form a closed differential system when grouped appropriately:

$$\dot{y} < 3y^{\frac{5}{4}} p - y^2, \quad \dot{y} > \frac{12}{5} y^3 v - 3y^2, \quad \dot{y} > y^2 - \frac{3}{w},$$

$$\dot{z} < 3z^2 - 3yz, \quad \dot{z} > \frac{6}{5} z^2 - 12yz, \quad \dot{w} < -6zw.$$

$$\dot{w} > -6zw - \sqrt{6} yw, \quad \dot{v} < -3\frac{z^2}{y^2} - \frac{z}{y} \quad \text{and} \quad \dot{p} > \frac{1}{10} p\,(6z - 107y)\,.$$

Data:

$$y(0) = \beta, \quad z(0) = \beta^3, \quad w(0) = 1, \quad v(0) = \beta \quad \text{and} \quad p(0) = \beta^{\frac{11}{4}}.$$

Estimation Chain (using comparison theorems)

1. Amplifying inequality, decouple out z.

$$\dot{z} < 3z^2 - \underbrace{3yz}_{>0} < 3z^2 \overset{\text{Riccati}}{\Longrightarrow} z > \frac{\beta^3}{-3\beta^3 t + 1}$$

2. Lower bound for $z \Rightarrow$ lower bound for w (ϕ).

$$\dot{w} < -6zw = -\frac{6\beta^3}{-3\beta^3 t + 1} w \implies w = e^{\phi} > (-3\beta^3 t+1)^2 \implies \phi > \ln(-3\beta^3 t+1)^2$$

3. Lower bound for $w \Rightarrow$ upper bound (I) for y (H).

$$\dot{y} > y^2 - \frac{3}{w} > y^2 - \frac{3}{(-3\beta^3 t + 1)^2} \implies y < \frac{\gamma}{-3\beta^3 t + 1} < \gamma$$

4. Upper bound for y + lower bound for $z \Rightarrow$ lower bound for v.

$$\dot{v} < -3\frac{z^2}{y^2} - \frac{z}{y} < -\frac{3\beta^6}{\gamma^2} - \frac{\beta^3}{\gamma} \implies v(t) > \beta - \theta t$$

5. Lower bound for v + upper bound (I) for y $\Rightarrow$ additional upper bound (II) for y. $y < \min\{(\mathrm{I}),(\mathrm{II})\}$

$$y^{-4}\dot{y} > \frac{12}{5}y^{-4}y^3 v - 3y^{-4}y^2 \overset{q=y^{-2}}{\Longrightarrow} -\frac{1}{2y}\dot{q} > \frac{12}{5}\frac{v}{y} - 3q$$

$$\Longrightarrow \dot{q} < -\frac{24}{5}v + 6yq < -\frac{24}{5}(\beta - \theta t) + 6\gamma q$$

$$\Longrightarrow \dot{q} - 6\gamma q < -\frac{24}{5}(\beta - \theta t) \leftarrow \boxed{\text{linear eq./Solved by integrating factor}}$$

$$\Longrightarrow q > e^{6\gamma t}\left(\frac{1}{\beta^2} - \frac{4\beta}{5\gamma} + \frac{2\theta}{15\gamma^2}\right) - \frac{2(-6\beta\gamma + \theta + 6\gamma\theta t)}{15\gamma^2}$$

$$\Longrightarrow y < \left(e^{6\gamma t}\left(\frac{1}{\beta^2} - \frac{4\beta}{5\gamma} + \frac{2\theta}{15\gamma^2}\right) - \frac{2(-6\beta\gamma + \theta + 6\gamma\theta t)}{15\gamma^2}\right)^{-\frac{1}{2}}$$

6. Upper bound for y $\Rightarrow$ upper bound for z.

$$\dot{z} > \frac{6}{5}z^2 - 12yz > -12yz > -\frac{12\gamma z}{-3\beta^3 t + 1} \Longrightarrow z(t) < \beta^3\left(-3\beta^3 t + 1\right)^{\frac{4\gamma}{\beta^3}}$$

7. Upper bound for z + Upper bound for y $\Rightarrow$ upper bound for w (ϕ).

$$\dot{w} > -6zw - \sqrt{6}yw > \left(-6\beta^3\left(-3\beta^3 t + 1\right)^{\frac{4\gamma}{\beta^3}} - \frac{\sqrt{6}\gamma}{-3\beta^3 t + 1}\right) w$$

$$\Longrightarrow \ w(t) < \left(-3\beta^3 t + 1\right)^{\frac{\sqrt{6}\gamma}{3\beta^3}} \exp\left(\frac{2\beta^3}{4\gamma + \beta^3}\left[\left(-3\beta^3 t + 1\right)^{\frac{4\gamma+\beta^3}{\beta^3}} - 1\right]\right)$$

$$\phi < \frac{\sqrt{6}\gamma}{3\beta^3}\ln(-3\beta^3 t + 1) + \left(\frac{2\beta^3}{4\gamma + \beta^3}\left[\left(-3\beta^3 t + 1\right)^{\frac{4\gamma+\beta^3}{\beta^3}} - 1\right]\right)$$

8. Lower bound for z + upper bound for y $\Rightarrow$ upper bound for p.

$$\dot{p} > \frac{1}{10}p\left(6z - 107y\right) > \frac{1}{10}p\left(\frac{6\beta^3}{-3\beta^3 t + 1} - \frac{107\gamma}{-3\beta^3 t + 1}\right)$$

$$\Longrightarrow \ p < \beta^{\frac{11}{4}}\left(-3\beta^3 t + 1\right)^{\frac{107\gamma}{30\beta^3} - \frac{1}{5}}$$

9. Upper bound for p $\Rightarrow$ lower bound for y (H).

$$\dot{y} < 3y^{\frac{5}{4}}p - y^2 < 3y^{\frac{5}{4}}\beta^{\frac{11}{4}}\left(-3\beta^3 t + 1\right)^{\frac{107\left(\sqrt{9\beta^6+12}+3\beta^3\right)}{60\beta^3} - \frac{1}{5}}$$

$$\Longrightarrow \ y(t) > \left[\beta^{-1/4} - \frac{\beta^{-1/4}}{4(m+1)}\left(1 - \left(-3\beta^3 t + 1\right)^{m+1}\right)\right]^{-4} > 0$$

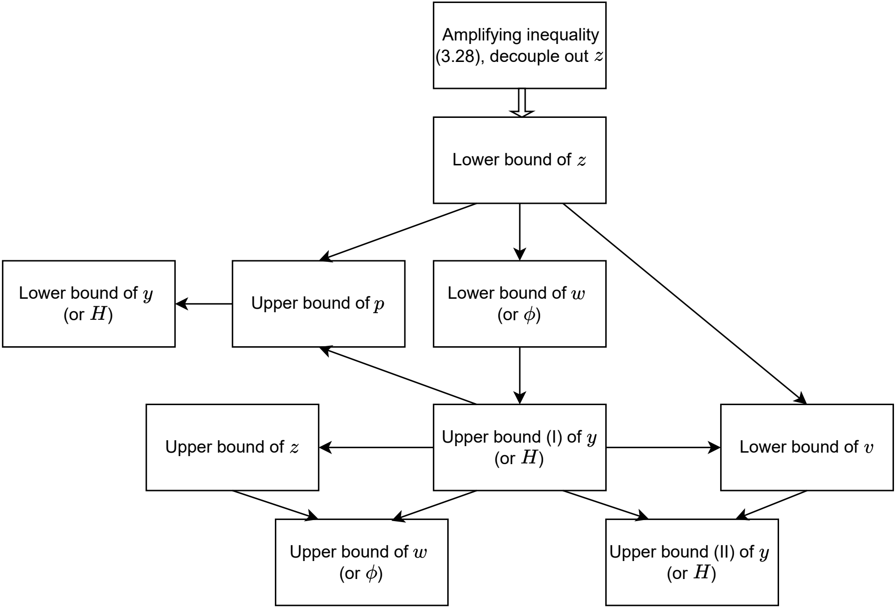


Figure: Hierarchical estimates.

Thank you for your attention!